

On the method of Bukhgeim for two-dimensional inverse problems

PhD Thesis

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En la presente tesis se considera el problema inverso de Gel'fand en el caso de dos dimensiones y se extiende el método de Bukhgeim para su resolución. El problema inverso de Gel'fand pertenece al campo de las ecuaciones en derivadas parciales y al análisis armónico y plantea la reconstrucción de un potencial desconocido dentro de un dominio a partir de mediciones realizadas en la frontera de dicho dominio. El método de Bukhgeim considera el uso de soluciones oscilatorias con frecuencia cuadrática para reconstruir el potencial a través de razonamientos similares al método de fase estacionaria para aproximar integrales oscilatorias.

El problema de Gel'fand, aparte de tener interés por sí mismo, es importante por su conexión con otros problemas inversos, por lo que su resolución permite abordar otra serie de preguntas. En esta tesis, además del problema de Gel'fand, también se abordan preguntas relacionadas con scattering inverso a energía fija y con el problema inverso de Calderón, donde se busca reconstruir la conductividad eléctrica de un objeto a partir de mediciones de voltajes y corrientes en la superficie del mismo.

De forma más precisa, en la presente tesis se obtienen los siguientes resultados de carácter novedoso:

- Reconstrucción de potenciales discontinuos en el problema de Gel'fand.
- Reconstrucción de potenciales discontinuos en scattering inverso a energía fija.
- Cota de estabilidad de potenciales discontinuos en el problema de Gel'fand.
- Cota de estabilidad de potenciales discontinuos en scattering inverso a energía fija.
- Reconstrucción de potenciales con mínima regularidad en el sentido de la norma de Sobolev L^2 en el problema de Gel'fand.
- Unicidad del problema inverso de Calderón para conductividades con valores complejos y regularidad Lipschitz.

Además, en la tesis también se exponen una serie de experimentos numéricos donde se estudia el desempeño de diferentes fórmulas de reconstrucción basadas en el método de Bukhgeim así como sus limitaciones.

Chapter 1

Introducción y conclusiones

1.1 Introducción

Considerese la ecuación de Schrödinger independiente del tiempo con potencial q en un dominio acotado Ω en \mathbb{R}^n . Dada una función f definida en la frontera, el problema de Dirichlet para esta ecuación es determinar el u que satisface

$$\begin{cases} \Delta u = qu & \text{en } \Omega \\ u|_{\partial\Omega} = f. \end{cases}$$

Bajo ciertas condiciones hay una única solución a este problema y podemos definir formalmente el mapa Dirichlet-to-Neumann (DtN) según

$$\Lambda_q[f] := \partial_\nu u|_{\partial\Omega}$$

donde ∂_ν denota la derivada exterior normal a la frontera. El problema inverso de Gel'fand consiste en recuperar el potencial q a partir del mapa DtN. Esto es, aplicando condiciones de Dirichlet en la frontera y midiendo la derivada exterior resultante, el objetivo es determinar q dentro de Ω .

Este problema surge de forma natural en la teoría de *scattering quantum*. En este contexto, el objetivo es calcular el potencial q a partir del conocimiento de como este distorsiona las ondas electromagnéticas. Las soluciones distorsionadas satisfacen la ecuación

$$-\Delta u + qu = k^2 u \quad \text{en } \mathbb{R}^n \tag{1.1}$$

donde k^2 es la energía total de la onda. Veremos como el conocimiento de la distorsión generada por el potencial a ondas con una determinada energía nos permite calcular el mapa DtN y, por tanto, este problema se puede reducir al problema de Gel'fand.

Este último problema está relacionado con un problema de scattering acústico, donde el objetivo es obtener información sobre un objeto (su

posición, densidad, etc.) a partir del conocimiento de cómo el objeto distorsiona las ondas acústicas que indican en él. En particular, sea $n(x)$ el índice de refracción del sonido dentro de un medio, tomando

$$q(x) = k^2(1 - n(x))$$

la ecuación (1.1) se cumple para ondas acústicas. Por tanto, el problema de scattering acústico inverso en un medio no homogéneo también se puede reducir al problema inverso de Gel'fand.

Finalmente, el problema de Gel'fand también está relacionado con el *problema inverso de la conductividad*, también conocido como el problema inverso de Calderón. Aquí el objetivo es calcular la conductividad eléctrica en cada punto en el interior de un dominio a partir de mediciones realizadas en la frontera del mismo. Sea Ω un dominio acotado y sea $\gamma(x)$ la conductividad dentro del mismo. El problema de Dirichlet para la ecuación de la conductividad es determinar v que satisface

$$\begin{cases} \nabla \cdot (\gamma \nabla v) = 0 & \text{en } \Omega \\ v|_{\partial\Omega} = f \end{cases}$$

dada una f definida en la frontera. Bajo ciertas condiciones, existe una solución única a este problema, y por tanto podemos definir el mapa DtN para este problema según

$$\Lambda_\gamma[v|_{\partial\Omega}] := \gamma \partial_\nu v|_{\partial\Omega}.$$

El problema inverso de la conductividad es calcular γ a partir de Λ_γ . Desde un punto de vista físico, estamos midiendo las corrientes eléctricas generadas a partir de los voltajes que imponemos en la frontera.

El problema inverso de la conductividad puede reducirse al problema inverso de Gel'fand, y esta reducción a sido frecuentemente utilizada para solucionar el primero. La relación es la siguiente: si v satisface

$$\nabla \cdot (\gamma \nabla v) = 0$$

entonces $u = \gamma^{1/2}v$ satisface la ecuación de Schrödinger

$$\Delta u = qu$$

donde el potencial viene dado por $q = \gamma^{-1/2}\Delta\gamma^{1/2}$.

Con respecto a las aplicaciones de estos problemas, que no se abordan en la tesis, múltiples modelos de scattering inverso aparecen en la física para poder observar distintos fenómenos. El scattering cuántico aparece en la física de partículas, donde se estudia la interacción entre partículas subatómicas. La teoría de scattering clásico también presenta multitud de aplicaciones, que van desde el sonar y el radar, a aplicaciones en medicina, la exploración geofísica o al testeo no destructivo en la industria.

Con respecto al problema inverso de la conductividad, seguramente la aplicación más relevante es la *tomografía por impedancia eléctrica*, un método no invasivo para generar imágenes médicas. Se colocan una serie de electrodos sobre la piel del paciente con los cuales se generan mediciones eléctricas, permitiendo obtener una imagen del interior del cuerpo del paciente. Esta técnica ha probado su utilidad en la monitorización de la actividad pulmonar y en la detección del cáncer de mama, y también se están considerando aplicaciones para realizar análisis del interior del cerebro (por ejemplo para la detección de isquemias y hemorragias).

1.2 Conclusiones

La presente tesis extiende el enfoque basado en soluciones cuadráticas introducido por Bukhgeim en [21] para resolver el problema inverso de la ecuación de Schrödinger en el plano. Basandonos en este procedimiento, hemos sido capaces de dar respuesta a la pregunta de cómo recuperar potenciales discontinuos y con valores complejos, hemos sido capaces de cuantificar la estabilidad de dicha reconstrucción, hemos probado que el problema inverso de Calderón con conductividades complejas y con regularidad Lipschitz tiene solución única, hemos sido capaces de desarrollar nuevas fórmulas de reconstrucción que permiten recuperar potenciales complejos pertenecientes a H^s para cualquier valor $s > 0$ y hemos comprobado como estas nuevas fórmulas son útiles desde la perspectiva de la implementación numérica de un algoritmo de reconstrucción. Además, presentamos los experimentos numéricos realizados y discutimos una serie de conclusiones obtenidas a partir de los mismos.

El capítulo 3 contiene trabajo previo que nos sirve para desarrollar nuestros resultados. Describimos como el problema de Gel'fand y el problema de scattering inverso a energía fija están conectados y también describimos el método de Bukhgeim para atacar al primero de ellos. También introducimos algunos de los resultados de Astala, Faraco y Rogers [7] que serán utilizados en los siguientes capítulos.

En el capítulo 4 consideramos potenciales con valores complejos en el plano que presentan discontinuidades sobre curvas. Vemos que una variante de la fórmula de reconstrucción de Bukhgeim, introducida por Astala, Faraco y Rogers en [7], puede utilizarse para recuperar los valores del potencial en casi todo punto en el interior del dominio a partir del conocimiento del mapa DtN. Apoyandonos en este resultado de reconstrucción, somos capaces de proporcionar una cota de estabilidad de tipo logarítmico en la norma L^∞ dado un conocimiento de la ubicación de las discontinuidades. También proporcionamos resultados similares para el problema de scattering inverso a energía fija, reconstruyendo potenciales con valores reales a partir de la amplitud de scattering. Los resultados contenidos en este capítulo han sido

publicados en [62].

En el capítulo 5 proporcionamos dos fórmulas de reconstrucción para el problema de Gel'fand en el plano basandonos en el enfoque de Bukhgeim. Estas nuevas fórmulas se basan en procedimientos de promediado que son útiles en dos direcciones. Por un lado, nos permite recuperar potenciales con valores complejos que pertenecen a H^s para cualquier $s > 0$; es significativo que el enfoque estándar se sabe que falla para potenciales que pertenecen a H^s con $s < 1/2$. En la otra dirección, estas fórmulas mejoran la tasa de convergencia con respecto a la fórmula original de Bukhgeim.

En el capítulo 6 probamos que las conductividades Lipschitz con valores complejos en el plano están unívocamente determinadas por el mapa DtN. La prueba se basa en la reducción de la ecuación de la conductividad a la ecuación de Dirac introducida por Brown y Uhlmann en [20] combinada con el método de Bukhgeim. Los resultados de este capítulo fueron publicados en un trabajo conjunto con Evgeny Lakshtanov y Boris Vainberg en [47] y se presentan en esta tesis con mayor detalle.

El capítulo 7 contiene experimentos numéricos realizados con el método de Bukhgeim. Estudiamos empíricamente la convergencia de la aproximación de fase estacionaria y la relacionamos con la propagación del error en las mediciones a la reconstrucción. También vemos como las fórmulas promediadas introducidas en la sección 5 mejoran significativamente la convergencia del término principal al potencial en distintos ejemplos.

Chapter 2

Introduction

2.1 Outline of the problems

Consider the time-independent Schrödinger equation with potential q in a bounded domain Ω in \mathbb{R}^n . Given a function f defined on the boundary, the Dirichlet problem for this equation is to determine u satisfying

$$\begin{cases} \Delta u = qu & \text{in } \Omega \\ u|_{\partial\Omega} = f. \end{cases}$$

Under certain conditions, there is a unique solution to this problem and we can formally define the Dirichlet to Neumann (DtN) map by

$$\Lambda_q[f] := \partial_\nu u|_{\partial\Omega}$$

where ∂_ν denotes the outward normal derivative on the boundary. The Gel'fand inverse problem consists of recovering the potential q from the DtN map. That is, by applying Dirichlet boundary conditions and measuring the resulting outward normal derivative, the goal is to estimate the values of q inside Ω .

This question arises naturally in *quantum scattering* theory. In this setting, the problem is to find the potential q from the knowledge of how it distorts incoming waves. The scattered waves are solutions to

$$-\Delta u + qu = k^2 u \quad \text{in } \mathbb{R}^n \tag{2.1}$$

where k^2 is the total energy of the wave. We will see how knowledge of scattering data at fixed energy yields knowledge of the DtN map, and so this problem can be reduced to the Gel'fand problem.

In fact this is related to an acoustic scattering problem, where the goal is to obtain information about an object (position, density, etc.) from the knowledge of how this object disturbs acoustic waves. In particular, letting $n(x)$ denote the refraction index of sound inside a medium, and taking

$$q(x) = k^2(1 - n(x))$$

equation (2.1) holds for acoustic waves. Thus, the acoustic inverse scattering problem in an inhomogeneous medium can also be reduced to the Gel'fand problem.

Finally, the Gel'fand problem is also linked with the *inverse conductivity problem*, also known as the Calderón inverse problem. Here the goal is to estimate the electric conductivity in each point in the interior of a domain from measurements performed on its surface. Let Ω be a bounded domain and let $\gamma(x)$ be the conductivity inside the domain. The Dirichlet problem for the conductivity equation is to determine v satisfying

$$\begin{cases} \nabla \cdot (\gamma \nabla v) = 0 & \text{in } \Omega \\ v|_{\partial\Omega} = f \end{cases}$$

given f defined on the boundary. Under certain conditions, there exists a unique solution to this problem, and so we can formally define the DtN map for this problem by

$$\Lambda_\gamma[v|_{\partial\Omega}] := \gamma \partial_\nu v|_{\partial\Omega}.$$

The inverse conductivity problem is to recover γ from the information contained in Λ_γ . From the physical point of view, we are measuring the currents induced by the voltages that we place on the surface.

The inverse conductivity problem can be reduced to Gel'fand problem, and this reduction has been extensively used for solving the former. The relation is the following: if v satisfies

$$\nabla \cdot (\gamma \nabla v) = 0$$

then $u = \gamma^{1/2}v$ solves the Schrödinger equation

$$\Delta u = qu$$

where the potential is given by $q = \gamma^{-1/2}\Delta\gamma^{1/2}$.

Regarding real-world applications, not touched on in this thesis, different inverse scattering models arise throughout physics in order to observe different phenomena. Quantum scattering appears in particle physics, where the interactions between subatomic particles are studied. Classical scattering theory also has broad applications, ranging from sonar and radar, to medical imaging, geophysical exploration or non-destructive testing.

As for the inverse conductivity problem perhaps the most relevant application is *electrical impedance tomography* (EIT), a non-invasive medical imaging technique. Electrodes are placed on the skin of a patient with which electric measurements are made and an image of the interior of the body can be constructed. This technique has proven to be useful in pulmonary imaging and in the detection of breast cancer, and applications in brain imaging have also been considered (e.g. for the detection of cerebral ischemia and haemorrhages).

2.2 Ill-posedness of the problems

Jacques Hadamard introduced the term of well-posed to refer to a problem that satisfied the following three conditions:

- There exists a solution to the problem.
- The solution is unique.
- The solution depends continuously on the initial data.

The inverse problems considered here are ill-posed. In particular, the conditions of uniqueness of the solution and continuity with respect to the initial data are only satisfied conditional to some a priori assumptions on the potential or conductivity.

Uniqueness is one of the most studied questions in the field. That is to say, proving that there is no more than one potential or conductivity that give rise to each DtN map or set of scattering data. One can consider for which class of conductivities and potentials uniqueness is guaranteed, but the opposite question is also interesting: which potentials or conductivities cannot be fully recovered from information on the boundary (*cloaking devices*).

With respect to the question of continuity, in inverse problems it is referred to as *stability*. This question has also been extensively studied in the literature. Logarithmic stability is often the best that can be hoped for.

Apart from the three conditions of well-posedness, there are a number of further interesting questions to address. Certainly one of the most relevant is the one of reconstruction: obtain a procedure for computing the potential or conductivity from the data at the boundary. This question is twofold, we can consider the task of creating a *theoretical* procedure for recovering the unknown coefficient or we may focus on creating a reconstruction procedure that works in *practice*, that is, a numerical procedure adapted to measurements on finitely many points and with finite precision.

Rather than a single and clear question to answer, the design and implementation of numerical reconstruction schemes is a field of research in its own right. Some of these numerical schemes are based on a theoretical reconstruction procedure, which frequently involve complex operations, such as singular integrals or the inversion of operators, whose numerical implementation is non-trivial and requires careful analysis. Other schemes are based on optimization techniques combined with regularization strategies.

2.3 History of the problems

The inverse boundary problem for the Schrödinger equation dates back to 1954, when it was proposed by Gel'fand in [33]. The inverse conductivity problem was considered first by Calderón in the 40's, even though his groundbreaking paper [22] was not published until 1980; here Calderón

proved uniqueness for the linearized problem by determining the Fourier transform of the conductivity. Both problems have been extensively studied since then, we address now only a few of the most relevant results.

In [45] Kohn and Vogelius proved uniqueness for analytic conductivities in dimension two or greater. They were able to recover the conductivity and its derivatives on the boundary from the DtN map and uniqueness followed from unique continuation. In [46] the same authors extended the result to deal with piecewise analytic conductivities.

In 1987 Sylvester and Uhlmann introduced the pionering work [61] where uniqueness for smooth conductivities and potentials is proved in dimension three or greater. Inspired by the original work of Calderón, the authors introduced complex geometric optics (CGO) solutions for the Schrödinger equation; these solutions have been extensively used since then. CGO solutions are parameterized by a complex vector $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and are of the form $u_\zeta(x) = e^{\zeta \cdot x}(1 + w_\zeta(x))$ where w_ζ is a remainder term which tends to zero in some sense as $|\zeta|$ grows. The power of these solutions relies on Alessandrini's identity; that

$$\int_{\partial\Omega} (\Lambda_q - \Lambda_0)[u] v = \int_{\Omega} q u v \quad (2.2)$$

whenever u is a solution to the Schrödinger equation and v is a solution to Laplace's equation. Then given any $\xi \in \mathbb{R}^n$ there exists a sequence of CGO solutions $\{\zeta_j\}$ such that $|\zeta_j| \rightarrow \infty$ and

$$\int_{\partial\Omega} (\Lambda_q - \Lambda_0)[u_{\zeta_j}] e^{\eta_j \cdot x} = \int_{\Omega} q e^{i\xi \cdot x} (1 + w_{\zeta_j})$$

where $\eta_j \in \mathbb{C}^n$ satisfies $\eta_j \cdot \eta_j = 0$ and $\zeta_j + \eta_j = \xi$. As the remainder terms w_{ζ_j} tend to zero, CGO solutions allow to determine the Fourier transform of the potential from the DtN map. This uniqueness result was refined later by Nachman, Sylvester and Uhlmann [52] to bounded potentials and by Brown [18] to deal with conductivities in $C^{1,1/2+\epsilon}$. Nachman [50] (parallel to a similar work by Novikov [53]) extended the uniqueness result to a reconstruction procedure.

Also relying on the scheme introduced in [61] is the remarkable article [36], where Haberman and Tataru proved uniqueness for Lipschitz conductivities satisfying $\|\nabla \log \gamma\|_{L^\infty} < \epsilon$ and for C^1 conductivities in dimensions three or greater. The authors introduce a new space of functions, inspired in Bourgain spaces, and an estimate of the decay for the remainder of the CGO solutions *on average*. This work has later been extended by Habermann [35] to prove uniqueness for conductivities in $W^{1,d}$ for $d = 3, 4$ and by Caro and Rogers [24], where unique determination for general Lipschitz conductivities in dimensions three or greater is proved, as conjectured by Uhlmann [63].

The two-dimensional case differs significantly to the higher-dimensional one. For example, even though CGO solutions continue to be used in this setting, a first difficulty to emulate the strategy of Sylvester and Uhlmann is that, for an arbitrary frequency ξ , one cannot longer express it as $\xi = \zeta + \eta$ with $\zeta \cdot \zeta = \eta \cdot \eta = 0$. The work by Nachman [51] in 1996 supposed a major breakthrough in the inverse conductivity in the plane. In this paper, a *non-physical* scattering transform is introduced. This transform can be computed from the DtN map, and conductivities can be recovered from it using the $\bar{\partial}$ -method (initially introduced in [15] by Beals and Coifman). Using this procedure a reconstruction scheme for conductivities with two derivatives in L^p with $p > 1$ is given. It is interesting that this scheme is designed only for potentials of *conductivity type* due to the existence of the so-called *exceptional points* for general potentials; these are values $\zeta \in \mathbb{C}^2$ satisfying $\zeta \cdot \zeta = 0$ for which the scattering transform cannot be defined due to the non-uniqueness of the CGO solutions. Brown and Uhlmann [20] extended this work, but instead of using the standard reduction to the Schrödinger equation, they reduced the conductivity equation to a first-order system (following Beals and Coifman [16]), which allowed them to prove unique determination of conductivities with one derivative (this work has been extended further in [31, 44]). Finally, in 2006 Astala and Päiväranta [11] extended the use of the scattering transform to solve the question of uniqueness for bounded conductivities and gave a reconstruction procedure. Their work rests on a reduction of the conductivity equation to a Beltrami equation combined with the theory of quasiconformal mappings.

In 2008 Bukhgeim [21] introduced a new method relying on a family of solutions which resemble CGO solutions but with quadratic phase. This new method allowed him to consider general potentials (that is, potentials which are not of conductivity type) in the two-dimensional setting, and he proved unique determination for complex-valued C^1 potentials. Blasten, Imanuvilov and Yamamoto [17] relaxed the smoothness assumption and proved unique determination for potentials in L^p with $p > 2$. Astala, Faraco and Rogers [7] extended the work of Bukhgeim to a reconstruction procedure for potentials in the plane with half a derivative in L^2 ; they also provide an example of a potential in $H^{1/2-\epsilon}$ which cannot be recovered on a set of positive measure using this procedure.

The study of the stability question was pioneered by the work of Alessandrini [2] in 1988, where logarithmic stability was proved for conductivities in $H^s(\mathbb{R}^n)$ with $s > 2 + n/2$ and $n \geq 3$. More precisely, he proved

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)})$$

where $\omega(t) \leq |\log(t)|^{-\delta}$ and δ depends only on n and s . This apparently weak estimate is in fact optimal, as was proved by Mandache [48]. The paper by Alessandrini also contains an example which shows that no L^∞ stability is

possible for discontinuous conductivities. Stefanov [59] obtained a stability estimate for the computation of the DtN map from the scattering amplitude, which allowed him to extend Alessandrini's work [2] to the inverse scattering problem with fixed energy data. In the recent paper [23] Caro, García and Reyes lowered the smoothness requirements and obtained a stability bound for conductivities in $C^{1,\epsilon}$ building on the work [36].

In two dimensions one of the first papers considering stability is [12], where Barceló, Barceló and Ruiz obtained a stability estimate for conductivities in $C^{1+\epsilon}$ relying on the reduction to the first order system introduced in [20]. The regularity assumption was later relaxed by Barceló, Faraco and Ruiz [14] and by Clop, Faraco and Ruiz [25] following the scheme introduced in [11]. Alessandrini and Cabib [5] provided counterexamples to L^2 stability for the conductivity equation based on G-Convergence which were systematically analyzed in [28] by Faraco, Kurylev and Ruiz; in [28] stability with respect to G-convergence it is also discussed. For general potentials in the plane, Novikov and Santacesaria [54] and Blasten, Imanuvilov and Yamamoto [17] obtained stability estimates using the quadratic phase approach introduced in [21]. Finally, in 2018 Faraco and Prats [29], extending the approach in [25], obtained a characterization of the sets of conductivities in the plane, in terms of the integral moduli of continuity, for which there exists stability in the L^2 norm, as conjectured by Alessandrini in [4].

In broad terms we could say that research focused on the development of numerical reconstruction schemes has taken two directions. On the one hand several authors have adapted theoretical reconstruction schemes to practice; here we can name the relevant works in two dimensions of [58] where Siltanen, Mueller and Isaacson gave a numerical implementation of the scheme in [51], [42, 43] where Knudsen implemented numerically the procedure in [20], or the work by Astala, Mueller, Päiväranta, Perämäki and Siltanen [9] where the procedure [11] is implemented numerically. The other direction of research is the adaptation of more general numerical techniques to this problem; see for example the books [26] and [40] for a general exposition of regularization and optimization strategies in inverse problems.

For the sake of brevity we have omitted other relevant questions, such as the study of inverse problems with partial data or more general settings, such as anisotropic conductivities or the Schrödinger equation with a magnetic term.

2.4 Outline of the thesis

The present thesis extends the quadratic phase approach introduced by Bukhgeim in [21] to address the inverse problem for the Schrödinger equation in the plane. Building on this procedure we are able to consider reconstruction and stability for discontinuous complex-valued potentials, we

prove unique determination of complex-valued Lipschitz conductivities and we develop some new reconstruction formulas which allow to recover H^s complex-valued potentials for any $s > 0$; these new formulas have also proven to be useful from a numerical point of view. We also present numerical experiments performed and discuss some conclusions derived from them.

Chapter 3 contains some previous work on which we rely on. We describe how the Gel'fand problem and inverse scattering at fixed energy are connected and we also describe the method of Bukhgeim to address the former. We will also introduce some of the results of Astala, Faraco and Rogers [7] which will be used in the following chapters.

In Chapter 4 we consider complex-valued potentials in the plane with discontinuities along curves. We show that a variant of Bukhgeim's reconstruction formula, introduced by Astala, Faraco and Rogers in [7], can be used to recover the values of the potential at almost every point inside the domain from the DtN map. Building on this reconstruction result, we are able to provide a logarithmic stability estimate in the L^∞ norm given an approximate knowledge of the location of the discontinuities. We also give similar results for inverse scattering with fixed energy, where real-valued potentials are reconstructed from the scattering amplitude. The results of this chapter were published in [62].

In Chapter 5 we provide two new reconstruction formulas for the Gel'fand problem in the plane based on the Bukhgeim approach. These new formulas rely on averaging procedures and are useful in two directions. On the one hand, we are able to recover complex-valued potentials in H^s for any $s > 0$; it is worth remarking that the standard approach is known to fail for potentials in H^s with $s < 1/2$. On the other hand, these formulas improve the rate of convergence over Bukhgeim's original formula.

In Chapter 6 we prove that complex-valued Lipschitz conductivities in the plane are uniquely determined by the DtN map. The proof relies on the reduction of the conductivity equation to the Dirac equation introduced by Brown and Uhlmann in [20] combined with the method of Bukhgeim. The results of this chapter were published in a joint work with Evgeny Lakshtanov and Boris Vainberg [47], and are presented here in more detail.

Chapter 7 contains some numerical experiments performed with the Bukhgeim method. We study empirically the convergence of the stationary phase approximation and relate its convergence with the propagation of measurement error to the reconstruction. We also show how the averaging formulas introduced in Section 5 significantly improve the convergence of the main term to the potential in a number of different examples.

Chapter 3

Preliminaries

3.1 Inverse scattering and the Gel'fand problem

Following [6], we reduce the fixed energy inverse scattering problem with compactly supported, real-valued potentials in $H^s(\mathbb{R}^2)$, with $s > 0$, to the Gel'fand problem.

First we introduce the inverse scattering problem at fixed energy. From the physical point of view, we are recovering a potential from the measurement of waves disturbed by the potential, where the incoming waves come from all directions, and the disturbed waves are measured in all directions. Let $k > 0$ and consider the Schrödinger equation

$$(\Delta + k^2)u = qu, \quad \text{in } \mathbb{R}^2, \quad (3.1)$$

where k^2 is not a Dirichlet eigenvalue for the Hamiltonian $-\Delta + q$. Let also

$$G_0(x, y) := -\frac{i}{4} H_0^{(1)}(k|x - y|)$$

where $H_n^{(1)}$ are the Hankel functions of the first kind. Then G_0 satisfies

$$(\Delta + k^2)G_0(x, y) = -\delta(x - y), \quad \forall x, y \in \mathbb{R}^2$$

as well as the outgoing Sommerfeld radiation condition

$$\lim_{|x| \rightarrow \infty} |x|^{1/2} \left(\frac{\partial}{\partial |x|} - ik \right) G_0(x, y) = 0; \quad (3.2)$$

see [57, Proposition 2.1]. For $\theta \in \mathbb{S}^1$ we can define the outgoing scattering solutions to (3.1) as the solutions to the Lippmann-Schwinger equation

$$u(x, \theta) := e^{ikx \cdot \theta} - \int_{\mathbb{R}^2} G_0(x, y) q(y) u(y, \theta) dy.$$

Then, the scattering amplitude $A_q : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{C}$ at energy k^2 satisfies

$$A_q(\eta, \theta) = \int_{\mathbb{R}^2} e^{-ik\eta \cdot y} q(y) u(y, \theta) dy.$$

Given an incident plane wave in direction θ , the scattering amplitude measures the probability of scattering in the direction η . The inverse scattering problem we consider is to compute q from A_q .

Now we introduce the Gel'fand problem in more detail. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Consider the following Dirichlet problem for the time-independent Schrödinger equation

$$\begin{cases} \Delta u = qu & \text{in } \Omega \\ u|_{\partial\Omega} = f. \end{cases} \quad (3.3)$$

Suppose that 0 is not a Dirichlet eigenvalue for the Hamiltonian $-\Delta + q$. Then for each $f \in H^{1/2}(\partial\Omega)$ there exists a unique solution $u \in H^1(\Omega)$ to (3.3) and the DtN map

$$\Lambda_q : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega) := (H^{1/2}(\partial\Omega))^*$$

can be defined via duality by

$$\langle \Lambda_q[f], v|_{\partial\Omega} \rangle = \int_{\Omega} q u v + \nabla u \cdot \nabla v \quad (3.4)$$

for any $v \in H^1(\Omega)$. Gel'fand's problem is to compute q from the knowledge of Λ_q . With sufficient regularity, the right-hand side of (3.4) equals

$$\int_{\partial\Omega} v (\nabla u \cdot \mathbf{n}),$$

by Green's identity, and so we see that this definition coincides with the one given in the introduction.

In order to relate the problems previously described, let G_q be the Green's function that satisfies

$$(-\Delta + q - k^2) G_q(x, y) = \delta(x - y), \quad \forall x, y \in \mathbb{R}^2,$$

and the outgoing Sommerfeld radiation condition given in (3.2). Using [6, Theorem 2.2] we can write G_q in terms of the scattering amplitude as

$$\begin{aligned} G_q(x, y) - G_0(x, y) &= \sum_{n, m \in \mathbb{Z}} \frac{(-1)^n}{16} i^{n+m} a_q^{(n, m)} \\ &\quad \times H_n^{(1)}(k|x|) H_m^{(1)}(k|y|) e^{in\alpha_x} e^{im\alpha_y} \end{aligned}$$

where $a_q^{(n,m)}$ are the Fourier coefficients of A_q

$$A_q(\eta, \theta) = \sum_{n,m \in \mathbb{Z}} a_q^{(n,m)} e^{in\eta} e^{im\theta}$$

and α_x is the angular coordinate of x . We can now define the single layer operator $\mathcal{S}_q : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ by

$$\mathcal{S}_q[f](x) := \int_{\partial\Omega} G_q(x, y) f(y) dy$$

which is invertible, see [39, Proposition A.1]. Then the problem of reconstruction from the scattering amplitude can be reduced to Gel'fand's problem using Nachman's formula

$$\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2} = \mathcal{S}_{q_1}^{-1} - \mathcal{S}_{q_2}^{-1},$$

taking $q_2 = 0$, see [50, Theorem 1.6].

3.2 Quadratic phase solutions

In this section we describe the fundamental aspects of Bukhgeim's method to attack Gel'fand's inverse problem. We also describe the variant of the reconstruction procedure due to Astala, Faraco and Rogers [7]; a number of their results will be useful in Chapter 4 and Chapter 5.

In [21] Bukhgeim considered solutions to the time-independent Schrödinger equation of the form

$$u_{\lambda,x} := e^{i\lambda\psi_x} (1 + w_{\lambda,x}), \quad \psi_x(z) := \frac{1}{2} (z_1 - x_1 + i(z_2 - x_2))^2, \quad (3.5)$$

where $w_{\lambda,x}$ tends to zero in some norm as λ tends to infinity. Given the value of these solutions at the boundary, the following recovery formula for smooth enough potentials holds:

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda}{\pi} \left\langle (\Lambda_q - \Lambda_0)[u_{\lambda,x}], e^{i\lambda\bar{\psi}_x} \right\rangle = q(x). \quad (3.6)$$

To construct the quadratic phase solutions, we use Wirtinger derivatives

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2), \quad \text{where } \partial_j = \frac{\partial}{\partial x_j},$$

to rewrite the Schrödinger equation as

$$4\partial\bar{\partial}u = qu.$$

Plugging-in Bukhgeim's solutions and multiplying both sides by $e^{i\lambda\bar{\psi}}$, this becomes

$$4e^{i\lambda\bar{\psi}}\partial\bar{\partial}e^{i\lambda\psi}(1+w) = e^{i\lambda\phi}q(1+w)$$

where $\phi = \phi_x$ is given by

$$\phi_x(z) := \psi_x(z) + \overline{\psi_x}(z) = (z_1 - x_1)^2 - (z_2 - x_2)^2.$$

Taking into account that $\bar{\partial}e^{i\lambda\psi} = \partial e^{i\lambda\bar{\psi}} = 0$, we find that

$$4\partial e^{i\lambda\phi}\bar{\partial}w = e^{i\lambda\phi}q(1+w).$$

As the derivatives are local operators and we only need to satisfy the equation inside Ω , we look for solutions that satisfy

$$w = \frac{1}{4}\bar{\partial}^{-1} \left[e^{-i\lambda\phi} \chi_Q \partial^{-1} \left[e^{i\lambda\phi} \chi_Q q(1+w) \right] \right]$$

where $\bar{\partial}^{-1}$ and ∂^{-1} denote the Cauchy transforms and Q is an auxiliary axis-parallel square containing Ω . In order to simplify notation we define the multiplication operators

$$M^{\pm\lambda}[f] = e^{\pm i\lambda\phi} \chi_Q f,$$

and write

$$S_1^\lambda = \frac{1}{4}\bar{\partial}^{-1} \circ M^{-\lambda} \circ \partial^{-1} \circ M^\lambda, \quad S_q^\lambda[f] = S_1^\lambda[qf].$$

For sufficiently smooth q , the operator norm of S_q^λ is small for large enough λ so we can invert $(I - S_q^\lambda)$ using Neumann series, yielding

$$w = (I - S_q^\lambda)^{-1} S_1^\lambda[q]. \quad (3.7)$$

In order to study the behaviour of these operators we will use the homogeneous L^2 Sobolev space, denoted by \dot{H}^s , with norm $\|f\|_{\dot{H}^s} = \| |\cdot|^s \hat{f} \|_{L^2}$, where the Fourier transform of f is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^2} f(x) e^{-2\pi i x \cdot \xi} dx.$$

The contraction required to compute w through Neumann series in (3.7) is obtained in three steps: first a contraction for $M^{\pm\lambda}$ is obtained in Lemma 3.2.1, this leads to the contraction for S_1^λ in Lemma 3.2.2, yielding the desired contraction for S_q^λ in Lemma 3.2.3.

The following lemma is a consequence of van der Corput's lemma.

Lemma 3.2.1. [7, Lemma 2.1] *Let $0 \leq s_1, s_2 < 1$. Then*

$$\left\| M^{\pm\lambda}[f] \right\|_{\dot{H}^{-s_2}} \leq C \lambda^{-\min\{s_1, s_2\}} \|f\|_{\dot{H}^{s_1}}, \quad \lambda \geq 1.$$

The following lemma was essentially proved in [7]; we present minor modifications, suitable for the stability analysis in Section 4.4.

Lemma 3.2.2. *Let $0 < s_1, s_2 < 1$. Then there exists a constant C such that*

$$\left\| S_1^\lambda[f] \right\|_{\dot{H}^{s_2}} \leq C \lambda^{-\tau} \|f\|_{\dot{H}^{s_1}}, \quad \lambda \geq 1,$$

where $\tau = 1 - s_2 + \min\{s_1, s_2\}$.

Proof. Using Lemma 3.2.1 twice we get

$$\begin{aligned} \left\| S_1^\lambda \right\|_{\dot{H}^{s_1} \rightarrow \dot{H}^{s_2}} &\leq \left\| M^{-\lambda} \circ \partial^{-1} \circ M^\lambda \right\|_{\dot{H}^{s_1} \rightarrow \dot{H}^{s_2-1}} \\ &\leq C \lambda^{-1+s_2} \left\| \partial^{-1} \circ M^\lambda \right\|_{\dot{H}^{s_1} \rightarrow \dot{H}^{1-s_2}} \\ &\leq C \lambda^{-1+s_2} \left\| M^\lambda \right\|_{\dot{H}^{s_1} \rightarrow \dot{H}^{-s_2}} \\ &\leq C \lambda^{-\tau} \end{aligned}$$

and the proof is concluded. \square

A consequence of Lemma 3.2.1 is the following decay estimate.

Lemma 3.2.3. [7, Lemma 2.3] *Let $0 < s < 1$. Then*

$$\left\| S_q^\lambda[f] \right\|_{\dot{H}^s} \leq C \lambda^{-\min\{2s, 1-s\}} \|q\|_{\dot{H}^s} \|f\|_{\dot{H}^s}, \quad \lambda \geq 1.$$

We now introduce the oscillatory integral operator T_g^λ defined by

$$T_g^\lambda[f](x) := \frac{\lambda}{\pi} \int_{\mathbb{R}^2} e^{i\lambda\phi_x(z)} f(z) g_{\lambda,x}(z) dz \quad (3.8)$$

and write $T^\lambda[f] := T_1^\lambda[f]$. The key of the reconstruction formula (3.6) is to use Alessandrini's identity

$$\begin{aligned} \frac{\lambda}{\pi} \left\langle (\Lambda_q - \Lambda_0)[u_{\lambda,x}], e^{i\lambda\bar{\psi}} \right\rangle &= \int_{\Omega} e^{i\lambda\phi_x} q (1 + w_{\lambda,x}) \\ &= T^\lambda[q](x) + T_w^\lambda[q](x). \end{aligned}$$

Then, on the one hand, the first term converges to reasonably smooth potentials by the following lemma.

Lemma 3.2.4. [7, Lemma 4.2] *Let $q \in \dot{H}^s(\mathbb{R}^2)$ with $1 < s < 3$. Then*

$$\left| T^\lambda[q] - q \right| \leq C \lambda^{\frac{1-s}{2}} \|q\|_{\dot{H}^s}.$$

On the other hand the second term tends to zero by the following lemma, the proof of which is essentially taken from [7] with minor modifications suitable for the stability analysis in Section 4.4.

Lemma 3.2.5. *Let $F, q \in \dot{H}^s(\mathbb{R}^2)$ where $0 < s < 1$. Then there exists a constant C such that*

$$\left| T_w^\lambda[f] \right| \leq C \lambda^{-s} \|f\|_{\dot{H}^s} \|q\|_{\dot{H}^s}$$

when λ is sufficiently large.

Proof. Using Lemma 3.2.1 we obtain

$$\begin{aligned} \left| T_w^\lambda[F](x) \right| &\leq C \lambda \left\| M^\lambda[F] \right\|_{\dot{H}^{-s}} \|w\|_{\dot{H}^s} \\ &\leq C \lambda^{1-s} \|F\|_{\dot{H}^s} \left\| (I - S_q^\lambda)^{-1} S_1^\lambda[q] \right\|_{\dot{H}^s}. \end{aligned}$$

As $(I - S_q^\lambda)^{-1}$ is bounded for λ sufficiently large by Lemma 3.2.3,

$$\begin{aligned} \left| T_w^\lambda[F](x) \right| &\leq C \lambda^{1-s} \|F\|_{\dot{H}^s} \left\| S_1^\lambda[q] \right\|_{\dot{H}^s} \\ &\leq C \lambda^{-s} \|F\|_{\dot{H}^s} \|q\|_{\dot{H}^s}, \end{aligned}$$

where the last inequality comes from applying Lemma 3.2.2. \square

The reconstruction formula (3.6) requires the values of the Bukhgeim solutions at the boundary. The following theorem allows to compute them from the DtN map.

Theorem 3.2.6. [7, Theorem 1.1] *Let $q \in H^s(\mathbb{R}^2)$ with $s > 0$ and suppose that Ω is Lipschitz. Then, for sufficiently large λ , we can identify compact operators $\Gamma_{\lambda,x} : H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$, depending only on λ, x and $\Lambda_q - \Lambda_0$, such that*

$$u_{\lambda,x}|_{\partial\Omega} = (I - \Gamma_{\lambda,x})^{-1} [e^{i\lambda\psi_x}|_{\partial\Omega}].$$

The operator $\Gamma_{\lambda,x}$ is defined by

$$\Gamma_{\lambda,x}[f] = T_\Omega [\langle (\Lambda_q - \Lambda_0)[f], G_{\lambda,x} \rangle]$$

where the Green's function $G_{\lambda,x}$ is given by

$$G_{\lambda,x}(z, \eta) = \chi_Q(\eta) \frac{e^{i\lambda(\psi_x(z) + \overline{\psi_x}(\eta))}}{4\pi^2} \int_Q \frac{e^{-i\lambda\phi_x(y)}}{(\overline{y} - \eta)(z - y)} dy$$

and where T_Ω is the trace operator on $\partial\Omega$ acting on the z variable.

This procedure for the construction of the solutions at the boundary is inspired by the one introduced by Nachman [50, 51] for linear phase solutions, however, there are significant differences between the two procedures. First of all, although $u_{\lambda,x}$ is defined for all $x \in \mathbb{R}^2$, it is not clear that they solve the Schrödinger equation globally. On the other hand, invertibility of the operator $\Gamma_{\lambda,x}$ stems from the contraction for S_q^λ , which ultimately comes from van der Corput's lemma for oscillatory integrals.

Chapter 4

Piecewise smooth potentials

4.1 Introduction

In this chapter we consider the two-dimensional inverse problem for the Schrödinger equation where the potentials are complex-valued and discontinuous on curves. This work was published in [62].

In Section 4.3 we obtain a reconstruction procedure from the DtN map. The precise statement is given in Theorem 4.3.3, where a decay rate for the error is also given. As a corollary we obtain recovery of real-valued potentials with discontinuities from the scattering amplitude. We also present a potential for which the recovery formula fails at points away from the discontinuities.

In Section 4.4 we give stability estimates from the DtN map and from the scattering amplitude. Notice that in [17] or in [25] there are stability estimates for discontinuous potentials but only in the L^2 sense. A careful analysis of the dependence of the constants in the reconstruction theorem yields an L^∞ stability estimate for discontinuous coefficients given a priori knowledge of the discontinuities of the potential. In particular, our Theorem 4.4.3 provides an estimate which offers better stability as the local regularity of the potentials increases.

4.2 Preliminaries

4.2.1 Piecewise $W^{s,1}$ -potentials

We say that a curve \mathcal{C} in the plane is contained in $C^{m,\alpha}$, with $m \geq 0$ and $0 \leq \alpha \leq 1$, if there exists a finite collection of bounded open sets $\{U_j\}_{j=1}^N$ such that $\mathcal{C} \subset \cup_{j=1}^N U_j$, and functions $f_j \in C^{m,\alpha}(\mathbb{R})$ such that

$$\mathcal{C} \cap U_j \subset \{(x, f_j(x)) : x \in \mathbb{R}\} \quad \text{or} \quad \mathcal{C} \cap U_j \subset \{(f_j(y), y) : y \in \mathbb{R}\}.$$

When f and its derivatives up to order m are continuous and bounded, occasionally we will describe the curve as being simply C^m .

Consider two curves \mathcal{C}_1 and \mathcal{C}_2 for which there is a finite cover by open sets $\{U_j\}_{j=1}^N$ such that for each j either we have

$$\mathcal{C}_1 \cap U_j \subset \{(x, f_{1,j}(x)) : x \in \mathbb{R}\} \quad \text{and} \quad \mathcal{C}_2 \cap U_j \subset \{(x, f_{2,j}(x)) : x \in \mathbb{R}\}$$

or we have

$$\mathcal{C}_1 \cap U_j \subset \{(f_{1,j}(y), y) : y \in \mathbb{R}\} \quad \text{and} \quad \mathcal{C}_2 \cap U_j \subset \{(f_{2,j}(y), y) : y \in \mathbb{R}\}.$$

Then we define the distance between the two curves in $C^{m,\alpha}$ norm as

$$d(\mathcal{C}_1, \mathcal{C}_2) = \inf \left\{ \sup_j \{ \|f_{1,j} - f_{2,j}\|_{C^{m,\alpha}} \} \right\}$$

where the infimum is taken over all possible common covers. If a common cover does not exist for the curves, we write $d(\mathcal{C}_1, \mathcal{C}_2) = \infty$.

Definition 4.2.1. *We say that q is a piecewise $W^{s,1}$ -potential whenever it can be expressed as*

$$q(x) = \sum_{j=1}^N q_j(x) \chi_{\Omega_j}(x),$$

where $q_j \in W^{s,1}(\mathbb{R}^2)$ and Ω_j are bounded Lipschitz domains whose boundaries are finite unions of $C^{2,\alpha}$ curves with $1/2 < \alpha \leq 1$.

The potentials that we consider are piecewise- $W^{s,1}$ with $2 \leq s < 3$, and therefore are potentials that exhibit line discontinuities. We will use the following norm for these potentials:

$$\|q\|_{D^{s,r}} = \inf \left\{ \sum_{j=1}^N \|q_j\|_{W^{s,1}} \left(1 + \|\chi_{\Omega_j}\|_{H^r} \right) : q(x) = \sum_{j=1}^N q_j(x) \chi_{\Omega_j}(x) \right\}$$

where $2 \leq s < 3, 0 < r < 1/2$ and q_j and Ω_j are as previously described.

The following lemma provides a bound for the L^2 Sobolev norm for the potentials of our interest.

Lemma 4.2.2. *Let $q : \mathbb{R}^2 \rightarrow \mathbb{C}$ and let Ω be a bounded Lipschitz domain in the plane. Then there exists a constant C independent of q and Ω such that*

i) *For $0 < r$ we have*

$$\|q\|_{H^r} \leq C \|q\|_{W^{r+1,1}}.$$

ii) *For $0 < r < 1/2$ we have*

$$\|q \chi_{\Omega}\|_{H^r} \leq C \|q \chi_{\Omega}\|_{D^{2,r}}.$$

Proof. Let m be the largest integer less than r , let $t = r - m$ and let $D^t = (-\Delta)^{t/2}$. For the first case we can use Sobolev embedding (see for example [1, Theorem 4.12, Part 1, Case C]) and the fact that $W^{r+1,1} \hookrightarrow W^{m+1,1}$ to obtain

$$\begin{aligned} \|q\|_{H^r} &\leq \left(\|q\|_{H^m}^2 + \|D^t q\|_{H^m}^2 \right)^{1/2} \leq C \left(\|q\|_{H^m}^2 + \|D^t q\|_{W^{m+1,1}}^2 \right)^{1/2} \\ &\leq C \left(\|q\|_{H^m}^2 + \|q\|_{W^{r+1,1}}^2 \right)^{1/2} \leq C \|q\|_{W^{r+1,1}}. \end{aligned}$$

For the second case we can use the generalized Leibniz rule [41, Theorem A.12], which states that

$$\|D^r(q \chi_\Omega) - q D^r(\chi_\Omega) - \chi_\Omega D^r(q)\|_{L^2} \leq C \|q\|_{L^\infty} \|D^r(\chi_\Omega)\|_{L^2},$$

and so by triangle inequality we obtain

$$\|q \chi_\Omega\|_{H^r} \leq C (\|q\|_{L^\infty} \|\chi_\Omega\|_{H^r} + \|q\|_{H^r} \|\chi_\Omega\|_{L^\infty}). \quad (4.1)$$

For the first term in the right-hand side, we can use Sobolev embedding (see for example [1, Theorem 4.12, Part 1, Case A]) to obtain

$$\|q\|_{L^\infty} \leq C \|q\|_{W^{2,1}}.$$

On the other hand, we have $\chi_\Omega \in H^r$ for $r < 1/2$; see for example [30]. For the second term in the right-hand side of (4.1) we can use case i , combined with the embedding $W^{2,1} \hookrightarrow W^{1+r,1}$, concluding the proof. \square

4.2.2 A topological property of graphs of C^1 functions

Now we provide a simple continuity result that will be useful for characterizing the topological properties of the set of points where the reconstruction is not guaranteed as well as the continuity properties of the error bound of the reconstruction.

Lemma 4.2.3. *Let $f, g \in C^1[a, b]$ and $\{x_j\}_{j=1}^N \in (a, b)$ be such that $f(x_j) = 0$, $f'(x_j) \neq 0$ and $f(x) \neq 0$ for all $x \in [a, b] \setminus \{x_j\}_{j=1}^N$. Then, for any $\epsilon > 0$, there exists δ such that if $\|f - g\|_{C^1} < \delta$ then there exists $\{x_j^*\}_{j=1}^N \in (a, b)$ such that*

$$|x_j - x_j^*| < \epsilon,$$

with $g(x_j^) = 0$, $g'(x_j^*) \neq 0$ and $g(x) \neq 0$ for all $x \in [a, b] \setminus \{x_j^*\}_{j=1}^N$.*

Proof. Let $\eta = \min_j \{|f'(x_j)|\}$ and let r be such that $|f'(x)| > \eta/2$ for all $x \in \cup B_r(x_j)$. Let

$$\tau = \inf_{x \in [a, b] \setminus \cup B_r(x_j)} \{|f(x)|\}.$$

Then, for all g such that

$$\|f - g\|_{C^1} < \frac{1}{2} \min\{\tau, \eta/2\},$$

we have

$$|g(x)| > \tau/2, \quad \forall x \in [a, b] \setminus \cup_j B_r(x_j)$$

and we also have

$$|g'(x)| > \eta/4, \quad \forall x \in \cup_j B_r(x_j).$$

Now, as $\|f - g\|_{C^0} < \tau/2$, we know that for all $x \in [a, b] \setminus \cup B_r(x_j)$ we have $f(x)g(x) > 0$ (f and g have the same sign outside the balls $B_r(x_j)$), and so, by the intermediate value theorem, g must vanish in each of the balls $B_r(x_j)$. The fact that g only vanishes at a single point x_j^* in each of the balls is a consequence of the fact that g is monotonous inside them. As $|g'(x)| > \eta/4$ inside the balls, then, whenever

$$\|f - g\|_{C^0} < \epsilon \eta/4,$$

we have that

$$|x_j - x_j^*| < \epsilon.$$

Taking $\delta = \min\{\tau/2, \eta/4, \epsilon \eta/4\}$ concludes the proof. \square

4.2.3 One-dimensional oscillatory integrals

Let $g : \mathbb{R} \rightarrow \mathbb{R} \in C^n$. We say that x_s is a stationary point of g of order $m < n$ if $g^{(k)}(x_s) = 0$ for $1 \leq k \leq m$ and $g^{(m+1)}(x_s) \neq 0$. We say that g has stationary points if such points exist. The following lemma describes the behavior of a one-dimensional oscillatory integral with a C^2 phase when there are only a finite number of stationary points of order one.

Lemma 4.2.4. *Let $h \in W^{1,1}([a, b])$ and $g \in C^2([a, b])$ be such that g has only a finite number of stationary points of order at most one. Then there exists a constant C , independent of h and depending continuously on the C^2 norm of g , such that*

$$\left| \int_a^b e^{i\lambda g(x)} h(x) dx \right| \leq C \lambda^{-1/2} \|h\|_{W^{1,1}([a, b])}$$

for $\lambda > 1$.

Proof. Let $\{s_j\}_{j=1}^N$ denote the stationary points and $\delta = \min_j (|g''(s_j)|)$. Let ϵ be such that $|g''(x)| > \delta/2$ for all $x \in \cup_j U_j$, where $U_j = (u_j^d, u_j^u) = B_\epsilon(s_j) \cap [a, b]$. Then we can make use of a version of van der Corput's lemma (see [34, Corollary 2.6.8]), to obtain

$$\left| \int_{U_j} e^{i\lambda g(x)} h(x) dx \right| \leq 24 \left(\frac{\delta}{2} \right)^{-1/2} \lambda^{-1/2} \left(|h(u_j^u)| + \int_{U_j} |h'(x)| dx \right).$$

Now let $V_j = (v_j^d, v_j^u)$ denote each of the remaining segments of $[a, b]$, such that $\cup_j V_j = [a, b] \setminus \cup_j U_j$. Integrating by parts in each V_j we obtain

$$\int_{V_j} e^{i\lambda g(x)} h(x) dx = \frac{1}{i\lambda} \left[\frac{e^{i\lambda g(x)} h(x)}{g'(x)} \right]_{v_j^d}^{v_j^u} - \frac{1}{i\lambda} \int_{V_j} e^{i\lambda g(x)} \frac{d}{dx} \left(\frac{h(x)}{g'(x)} \right) dx.$$

By Sobolev embedding [1, Theorem 4.12, Part 1, Case A] we have that $\|h\|_{L^\infty[a,b]} \leq C \|h\|_{W^{1,1}[a,b]}$. Making use of this and Hölder's inequality, altogether we obtain

$$\begin{aligned} \left| \int_a^b e^{i\lambda g(x)} h(x) dx \right| &\leq 48 N \left(\frac{\delta}{2} \right)^{-1/2} \lambda^{-1/2} \|h\|_{W^{1,1}([a,b])} \\ &\quad + (N+1) \kappa \lambda^{-1} \|h\|_{W^{1,1}([a,b])}, \end{aligned}$$

where

$$\kappa = \max_j \left\{ 2 \|(g')^{-1}\|_{L^\infty(V_j)} + \left\| \frac{g' - g''}{(g')^2} \right\|_{L^\infty(V_j)} \right\},$$

which is finite, as there are no stationary points in $\cup_j V_j$. Taking

$$C = 100 N \delta^{-1/2} + (N+1) \kappa$$

the proof is done, as δ and κ depend continuously on the C^2 norm of g . \square

4.3 Recovery

Later we will see that recovery is not guaranteed even at points that lie far from the discontinuities of the potential. In order to bound the measure of these points we will require the following key lemmas.

Lemma 4.3.1. *Let \mathcal{C} be a C^1 curve contained in a bounded planar domain Ω . Then, the union of tangent lines to \mathcal{C} with a fixed slope s has zero Lebesgue measure in Ω .*

Proof. Let $\mathcal{C} = \{(x, \Gamma(x))\}$, $\epsilon > 0$ and

$$I_\epsilon = \{x : \Gamma(x) \in \Omega, |\Gamma'(x) - s| < \epsilon\}.$$

As $\Gamma \in C^1$, I_ϵ is open, and as the real line satisfies the countable chain condition, I_ϵ must consist of a countable union of disjoint intervals $\{U_j\}$. For $x_0, x_1 \in U_j$, $x_0 < x_1$, by the Fundamental Theorem of Calculus, we have

$$\Gamma(x_1) = \Gamma(x_0) + \int_{x_0}^{x_1} \Gamma'(x) dx.$$

As $|\Gamma'(x) - s| < \epsilon$ in the domain of integration, then

$$(x_1 - x_0)(s - \epsilon) < \Gamma(x_1) - \Gamma(x_0) < (x_1 - x_0)(s + \epsilon),$$

and

$$|\Gamma(x_1) - \Gamma(x_0) - s(x_1 - x_0)| < \epsilon(x_1 - x_0). \quad (4.2)$$

Now let $l_{x_0,s}$ denote the line-segment with slope s that contains the point $(x_0, \Gamma(x_0))$;

$$l_{x_0,s} = \{(x, y) \in \Omega : y = \Gamma(x_0) + s(x - x_0)\}.$$

We write $p_0 = (x_0, \Gamma(x_0)) \in l_{x_0,s}$ and $p_1 = (x_1, \Gamma(x_1)) \in l_{x_1,s}$. As $p_0 \in l_{x_0,s}$, then so is $p_0^* = (x_1, \Gamma(x_0) + s(x_1 - x_0))$. By (4.2) we know that $d(p_0^*, p_1) < \epsilon(x_1 - x_0)$ and so it follows that $L_j = \bigcup_{x \in U_j} l_{x,s}$ is contained in a rectangle of width bounded by $\epsilon|U_j|$ and length bounded by $\text{diam}(\Omega)$. Thus,

$$\left| \bigcup_j L_j \right| \leq \sum_j |L_j| < \sum_j \epsilon|U_j| \text{diam}(\Omega) \leq \epsilon \text{diam}(\Omega)^2.$$

Letting ϵ tend to zero, the proof is complete. \square

Recall that $\phi_x(z) = (z_1 - x_1)^2 - (z_2 - x_2)^2$.

Lemma 4.3.2. *Let Ω be a bounded domain in the plane and let \mathcal{C} be a curve contained in Ω which is the graph of a $C^{2,\alpha}$ function with $1/2 < \alpha \leq 1$. Then the set of points $x \in \Omega$ such that either*

i) $\phi_x|_{\mathcal{C}}$ has an infinite number of stationary points,

ii) $\phi_x|_{\mathcal{C}}$ has at least one stationary point of order greater than one,

has zero Lebesgue measure and is closed.

Proof. First we see that if $\phi_x|_{\mathcal{C}}$ has an infinite number of stationary points, it has at least one stationary point of order greater than one, and so the first case is contained in the second. Let x be such that $\phi_x|_{\mathcal{C}}$ has an infinite number of stationary points. Then, by the compactness of \mathcal{C} , there exists a sequence of stationary points $\{z_{1,i}\}_{i=1}^{\infty}$ and a point $z_{1,\infty}$ such that

$$\lim_{i \rightarrow \infty} z_{1,i} = z_{1,\infty}$$

with

$$\frac{\partial \phi_x|_{\mathcal{C}}}{\partial z_1}(z_{1,\infty}) = 0.$$

As $\phi_x|_{\mathcal{C}}$ is a C^2 function and $\frac{\partial \phi_x|_{\mathcal{C}}}{\partial z_1}$ vanishes at all the stationary points then

$$\frac{\partial^2 \phi_x|_{\mathcal{C}}}{\partial z_1^2}(z_{1,\infty}) = \lim_{i \rightarrow \infty} \frac{\frac{\partial \phi_x|_{\mathcal{C}}}{\partial z_1}(z_{1,i+1}) - \frac{\partial \phi_x|_{\mathcal{C}}}{\partial z_1}(z_{1,i})}{z_{1,i+1} - z_{1,i}} = 0,$$

and therefore $z_{1,\infty}$ is a stationary point of order greater than one.

Now we see that the set of x such that $\phi_x|_{\mathcal{C}}$ has a stationary point of order greater than one is null. That is, the set of x such that

$$\frac{\partial \phi_x|_{\mathcal{C}}}{\partial z_1} = \frac{\partial^2 \phi_x|_{\mathcal{C}}}{\partial z_1^2} = 0$$

has zero measure. Letting $\Gamma \in C^{2,\alpha}$ be such that $\mathcal{C} = \{(z_1, \Gamma(z_1))\}$, the previous condition can be written as

$$z_1 - x_1 - \Gamma'(z_1)(\Gamma(z_1) - x_2) = 1 - \Gamma''(z_1)(\Gamma(z_1) - x_2) - \Gamma'(z_1)^2 = 0$$

leading to

$$x_1 = z_1 + \Gamma'(z_1)(x_2 - \Gamma(z_1)), \quad (4.3)$$

$$\Gamma''(z_1)(x_2 - \Gamma(z_1)) = \Gamma'(z_1)^2 - 1. \quad (4.4)$$

First we consider the case where $|\Gamma''(z_1)| > \delta > 0$. As Γ'' is continuous and the real line satisfies the countable chain condition, for this to be satisfied z_1 must lie in one of at most countably many intervals. Taking one such interval U and rearranging (4.3) and (4.4), the set of x such that $\phi_x|_{\mathcal{C}}$ has a stationary point of order greater than one at $z_1 \in U$ is given by

$$x = (x_1, x_2) = G(z_1) = \left(z_1 + \frac{\Gamma'(z_1)^3 - \Gamma'(z_1)}{\Gamma''(z_1)}, \Gamma(z_1) + \frac{\Gamma'(z_1)^2 - 1}{\Gamma''(z_1)} \right).$$

We see that the set of x is the image of a $C^{0,\alpha}$ function. To see that such a set has zero measure, take $\{U_j\}_{j=1}^{2N}$ a covering of U such that $|U_j| = |U|/N$. Now as $G(U_j)$ is contained in a ball of radius $\lesssim (|U|/N)^\alpha$, we obtain $|G(U)| \lesssim |U|^{2\alpha} N^{1-2\alpha}$. As $\alpha > 1/2$, we can let N tend to infinity to conclude that $|G(U)| = 0$. This is the only place where we require the Hölder regularity. Now as the countable union of null sets is null, we have concluded the proof in this case.

On the other hand, if $|\Gamma''(z_1)| \leq \delta$ and $x \in \Omega$, then by (4.4) it follows that $\Gamma'(z_1)^2$ must be contained in the interval $[1 - \delta^*, 1 + \delta^*]$ with $\delta^* = \delta \text{diam}(\Omega)$. Let $l_{z_1,s}$ be the line that passes through $(z_1, \Gamma(z_1))$ with slope s , and let

$$T(\delta^*) = \bigcup l_{z_1, 1/\Gamma'(z_1)}, \quad \forall z_1 : \Gamma'(z_1)^2 \in [1 - \delta^*, 1 + \delta^*].$$

From equation (4.3) we see that the remaining set of x such that $\phi_x|_{\mathcal{C}}$ has a stationary point of order greater than one at $(z_1, \Gamma(z_1))$ is contained in $T(\delta^*)$. Using Lemma 4.3.1, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} |T(1/n) \cap \Omega| &= \left| \bigcap_{n=1}^{\infty} (T(1/n) \cap \Omega) \right| = |T(0) \cap \Omega| \\ &= \left| \bigcup_{z_1: \Gamma'(z_1)=1} l_{z_1,1} \cap \Omega \right| + \left| \bigcup_{z_1: \Gamma'(z_1)=-1} l_{z_1,-1} \cap \Omega \right| = 0. \end{aligned}$$

Therefore, for any $\varepsilon > 0$ we can take δ small enough so that $|T(\delta^*) \cap \Omega| < \varepsilon$, allowing us to conclude that the set of points x such that $\phi_x|_C$ has a stationary point of order greater than one is null.

To see that the set is closed, first notice that for any $\delta > 0$, there exists $r > 0$ such that for any $x' \in B_r(x)$ we have

$$\|\phi_x|_C - \phi_{x'}|_C\|_{C^2} < \delta.$$

Thus, applying Lemma 4.2.3 to $\frac{\partial \phi_x|_C}{\partial z_1}$ we see that whenever $\phi_x|_C$ has a finite number of stationary points of degree at most one then $\phi_{x'}|_C$ has the same number of stationary points and of the same degree for any x' close enough to x . This means that the set of points x such that $\phi_x|_C$ has a finite number of stationary points of degree at most one is open, and the complement is closed, concluding the proof. \square

Let $u_{\lambda,x}$ be the Bukhgeim solutions given in (3.5). Our reconstruction theorem for discontinuous potentials is the following.

Theorem 4.3.3. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^2 , let q be a piecewise $W^{s,1}$ -potential, with $0 < s - 2 < 2r < 1$, and let $u_{\lambda,x}$ be Bukhgeim solutions to $\Delta u = qu$. Then, for almost every $x \in \Omega$, there exists a constant $C_x = C(x, \cup \partial\Omega_j)$ such that*

$$\left| \frac{\lambda}{\pi} \int_{\partial\Omega} e^{i\lambda\bar{\psi}} (\Lambda_q - \Lambda_0) [u_{\lambda,x}] - q(x) \right| \leq C_x \lambda^{1-s/2} \left(\|q\|_{D^{s,r}} + \|q\|_{D^{s,r}}^2 \right)$$

whenever λ is sufficiently large. Moreover, if $s = 2$ then we have

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda}{\pi} \int_{\partial\Omega} e^{i\lambda\bar{\psi}} (\Lambda_q - \Lambda_0) [u_{\lambda,x}] = q(x), \quad \text{a.e. } x \in \Omega.$$

Proof. As the DtN is a self-adjoint operator and $e^{i\lambda\bar{\psi}_x}$ satisfies the Laplace equation, then we can use the DtN map definition (3.4) to see that

$$\frac{\lambda}{\pi} \int_{\partial\Omega} e^{i\lambda\bar{\psi}_x} (\Lambda_q - \Lambda_0) [u] = \frac{\lambda}{\pi} \int_{\Omega} e^{i\lambda\phi_x} q (1 + w).$$

Recalling that $0 \leq s - 2 < 2r < 1$, by Lemma 3.2.5 we have

$$\sup_{x \in \Omega} \left| \frac{\lambda}{\pi} \int_{\Omega} e^{i\lambda\phi_x} q w \right| \leq C \lambda^{-r} \|q\|_{H^r}^2, \quad \text{as } \lambda \rightarrow \infty,$$

and by part *ii* of Lemma 4.2.2 this yields

$$\sup_{x \in \Omega} \left| \frac{\lambda}{\pi} \int_{\Omega} e^{i\lambda\phi_x} q w \right| \leq C \lambda^{1-s/2} \|q\|_{D^{s,r}}^2, \quad \text{as } \lambda \rightarrow \infty. \quad (4.5)$$

Now, by Lemma 4.3.2 we know that for almost every x in Ω , $\phi_x|_{\cup \partial\Omega_j}$ has only a finite number of stationary points of order at most one. We

now prove that the reconstruction formula recovers the potential correctly at these points for piecewise $W^{s,1}$ -potentials, $s > 2$ (almost all of them for $W^{2,1}$ -potentials). First we split the integral

$$\frac{\lambda}{\pi} \int_{\Omega} e^{i\lambda\phi_x} q = \frac{\lambda}{\pi} \sum_{j=1}^N \int_{\Omega_j} e^{i\lambda\phi_x} q_j, \quad (4.6)$$

where we write ϕ for ϕ_x from now on. We will prove that the value of each of these integrals tends to zero sufficiently fast whenever the integration domain does not contain x , the point at which we are reconstructing. Then we show that the value of the integral that contains x converges to $q(x)$. Without loss of generality, we can suppose that x belongs to the interior of Ω_1 . For $j > 1$, we use Green's first identity, with $u = \frac{e^{i\lambda\phi}}{i\lambda}$ and $\nabla v = \frac{q_j \nabla \phi}{\|\nabla \phi\|^2}$, to obtain

$$\int_{\Omega_j} e^{i\lambda\phi} q_j = \frac{1}{i\lambda} \int_{\partial\Omega_j} e^{i\lambda\phi} \frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \cdot \mathbf{n} - \frac{1}{i\lambda} \int_{\Omega_j} e^{i\lambda\phi} \nabla \cdot \left(\frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right).$$

Using Green's first identity again on the second term with $u = \frac{e^{i\lambda\phi}}{i\lambda}$ and $\nabla v = \nabla \cdot \left(\frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right) \frac{\nabla \phi}{\|\nabla \phi\|^2}$ leads to

$$\begin{aligned} \frac{\lambda}{\pi} \int_{\Omega_j} e^{i\lambda\phi} q_j &= \frac{1}{i\pi} \int_{\partial\Omega_j} e^{i\lambda\phi} \frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \cdot \mathbf{n} \\ &\quad + \frac{1}{\pi\lambda} \int_{\partial\Omega_j} e^{i\lambda\phi} \nabla \cdot \left(\frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right) \frac{\nabla \phi}{\|\nabla \phi\|^2} \cdot \mathbf{n} \\ &\quad - \frac{1}{\pi\lambda} \int_{\Omega_j} e^{i\lambda\phi} \nabla \cdot \left(\nabla \cdot \left(\frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right) \frac{\nabla \phi}{\|\nabla \phi\|^2} \right). \end{aligned} \quad (4.7)$$

As the number of stationary points on $\partial\Omega_j$ is finite and are of order at most one, and by trace theorem we know that $q_j|_{\partial\Omega_j} \in W^{1,1}(\partial\Omega_j)$ (see for example [27, Section 5.5, Theorem 1]), then we can use Lemma 4.2.4 on each of the C^2 components of $\partial\Omega_j$, together with Hölder's inequality, to see that there exists $C'_x = C(x, \partial\Omega_j)$ such that

$$\left| \int_{\partial\Omega_j} e^{i\lambda\phi} \frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \cdot \mathbf{n} \right| \leq C'_x \lambda^{-1/2} \left\| \frac{\nabla \phi \cdot \mathbf{n}}{\|\nabla \phi\|^2} \right\|_{W^{1,\infty}(\partial\Omega_j)} \|q_j\|_{W^{1,1}(\partial\Omega_j)}, \quad (4.8)$$

as $\lambda \rightarrow \infty$. As x belongs to the interior of Ω_1 , we have that $\|\nabla \phi\|^{-2}$ is bounded, and using the trace theorem this yields to

$$\left| \int_{\partial\Omega_j} e^{i\lambda\phi} \frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \cdot \mathbf{n} \right| \leq C_x \lambda^{-1/2} \|q\|_{D^{s,r}}, \quad \text{as } \lambda \rightarrow \infty. \quad (4.9)$$

For the second term on the right-hand side of (4.7) we can use Hölder's inequality to obtain

$$\begin{aligned}
& \left| \int_{\partial\Omega_j} e^{i\lambda\phi} \nabla \cdot \left(\frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right) \frac{\nabla \phi}{\|\nabla \phi\|^2} \cdot \mathbf{n} \right| \\
& \leq \left\| \nabla \cdot \frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right\|_{L^1(\partial\Omega_j)} \left\| \frac{\nabla \phi \cdot \mathbf{n}}{\|\nabla \phi\|^2} \right\|_{L^\infty(\partial\Omega_j)} \\
& \leq \left(\|\phi\|_{\dot{W}^{1,\infty}(\partial\Omega_j)} \|\|\nabla \phi\|^{-2}\|_{L^\infty(\partial\Omega_j)} + \left\| \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|^2} \right\|_{L^\infty(\partial\Omega_j)} \right) \\
& \quad \times \left\| \frac{\nabla \phi \cdot \mathbf{n}}{\|\nabla \phi\|^2} \right\|_{L^\infty(\partial\Omega_j)} \|q_j\|_{W^{1,1}(\partial\Omega_j)},
\end{aligned}$$

and by the trace theorem we get

$$\left| \int_{\partial\Omega_j} e^{i\lambda\phi} \nabla \cdot \left(\frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right) \frac{\nabla \phi}{\|\nabla \phi\|^2} \cdot \mathbf{n} \right| \leq C_x \|q\|_{D^{s,r}}. \quad (4.10)$$

Similarly, for the last term on the right-hand side of (4.7) we have

$$\begin{aligned}
& \left| \int_{\Omega_j} e^{i\lambda\phi} \nabla \cdot \left(\nabla \cdot \left(\frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right) \frac{\nabla \phi}{\|\nabla \phi\|^2} \right) \right| \\
& \leq \left\| \nabla \cdot \frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right\|_{\dot{W}^{1,1}(\Omega_j)} \|\phi\|_{\dot{W}^{1,\infty}(\Omega_j)} \|\|\nabla \phi\|^{-2}\|_{L^\infty(\Omega_j)} \\
& \quad + \left\| \nabla \cdot \frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right\|_{L^1(\Omega_j)} \left\| \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|^2} \right\|_{L^\infty(\Omega_j)} \\
& \leq \left(\|\phi\|_{W^{2,\infty}(\Omega_j)} \|\|\nabla \phi\|^{-2}\|_{W^{1,\infty}(\Omega_j)} + \left\| \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|^2} \right\|_{\dot{W}^{1,\infty}(\Omega_j)} \right) \\
& \quad \times \|\phi\|_{\dot{W}^{1,\infty}(\Omega_j)} \|\|\nabla \phi\|^{-2}\|_{L^\infty(\Omega_j)} \|q_j\|_{W^{2,1}(\Omega_j)} \\
& \quad + \left(\|\phi\|_{\dot{W}^{1,\infty}(\Omega_j)} \|\|\nabla \phi\|^{-2}\|_{L^\infty(\Omega_j)} + \left\| \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|^2} \right\|_{L^\infty(\Omega_j)} \right) \\
& \quad \times \left\| \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|^2} \right\|_{L^\infty(\Omega_j)} \|q_j\|_{W^{1,1}(\Omega_j)} \\
& \leq \left(\|\phi\|_{W^{2,\infty}(\Omega_j)} \|\|\nabla \phi\|^{-2}\|_{W^{1,\infty}(\Omega_j)} + \left\| \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|^2} \right\|_{W^{1,\infty}(\Omega_j)} \right)^2 \\
& \quad \times \|q_j\|_{W^{2,1}(\Omega_j)}
\end{aligned}$$

yielding

$$\left| \int_{\Omega_j} e^{i\lambda\phi} \nabla \cdot \left(\nabla \cdot \left(\frac{q_j \nabla \phi}{\|\nabla \phi\|^2} \right) \frac{\nabla \phi}{\|\nabla \phi\|^2} \right) \right| \leq C_x \|q\|_{D^{s,r}}. \quad (4.11)$$

Plugging (4.9), (4.10) and (4.11) into (4.7) we obtain

$$\left| \frac{\lambda}{\pi} \int_{\Omega_j} e^{i\lambda\phi} q_j \right| \leq C_x \lambda^{-1/2} \|q\|_{D^{s,r}} \quad \text{as } \lambda \rightarrow \infty. \quad (4.12)$$

We now consider $\int_{\Omega_1} e^{i\lambda\phi} q_1$ by decomposing q_1 into

$$q_x = q_1 \chi, \quad q_{\text{rem}} = q_1 (1 - \chi),$$

where $\chi(z)$ is a bump function such that

$$\chi(z) = \begin{cases} 1 & \text{if } \|z - x\| \leq r_1, \\ 0 & \text{if } \|z - x\| \geq r_2, \end{cases}$$

with $0 < r_1 < r_2 < d(x, \partial\Omega_1)$. As $q_{\text{rem}}(y) = 0$ for y close enough to x , we can use the same arguments that lead to (4.12) to obtain

$$\left| \frac{\lambda}{\pi} \int_{\Omega_1} e^{i\lambda\phi} q_{\text{rem}} \right| \leq C_x \lambda^{-1/2} \|q_{\text{rem}}\|_{W^{2,1}}, \quad \text{as } \lambda \rightarrow \infty. \quad (4.13)$$

On the other hand, as $q_1 \in W^{2,1}(\Omega_1)$, we can use Sobolev embedding (see for example [1, Theorem 4.12, Part 1, Case C]) to see that $q_x \in H_0^1(\Omega_1)$. Now, as was noted in [7], $\frac{\lambda}{\pi} \int_{\Omega_1} e^{i\lambda\phi} q_x$ can be interpreted as the solution to a nonelliptic time dependent Schrödinger equation at time $1/\lambda$. Thus, using the almost everywhere convergence result of [56, Theorem 1] we obtain

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda}{\pi} \int_{\Omega_1} e^{i\lambda\phi} q_x = q_x(x) = q_1(x) = q(x) \quad \text{a.e. } x \in \mathbb{R}^2. \quad (4.14)$$

If $q_1 \in W^{s,1}$ with $s > 2$, then we can recover at all the remaining points and we get a decay rate. Indeed, using Lemma 3.2.4 we have

$$\left| \frac{\lambda}{\pi} \int_{\Omega_1} e^{i\lambda\phi} q_x - q_x(x) \right| \leq C \lambda^{1-s/2} \|q_x\|_{\dot{H}^{s-1}}, \quad \text{as } \lambda \rightarrow \infty$$

and using part *i* of Lemma 4.2.2 yields

$$\left| \frac{\lambda}{\pi} \int_{\Omega_1} e^{i\lambda\phi} q_x - q_x(x) \right| \leq C \lambda^{1-s/2} \|q_x\|_{W^{s,1}}, \quad \text{as } \lambda \rightarrow \infty. \quad (4.15)$$

Plugging (4.12), (4.13), (4.14), and (4.15) into (4.6), together with (4.5) concludes the proof. \square

Remark 4.3.4. As noted in [7], $\frac{\lambda}{\pi} \int_{\Omega} e^{i\lambda\phi} q$ can be interpreted as the solution to a nonelliptic time-dependent Schrödinger equation at time $1/\lambda$. Therefore equations (4.12), (4.13), (4.14), and (4.15) imply almost everywhere convergence to the initial data q , whenever q is piecewise- $W^{s,1}$ with $2 \leq s < 3$.

A consequence of Theorem 4.3.3 is that potentials of this type can also be recovered from the scattering data at a fixed energy when these are real-valued.

Corollary 4.3.5. Let q be a real-valued piecewise $W^{2,1}$ -potential. Then q can be recovered almost everywhere from the scattering amplitude at a fixed energy k^2 .

Proof. Let Q be a square such that $\cup_{j=1}^N \Omega_j \subset Q$. Using the expressions in Section 3.1 we can compute Λ_{q-k^2} defined on ∂Q from the scattering amplitude at energy k^2 . Therefore, the recovery from the scattering amplitude follows directly from the fact that if q is piecewise $W^{2,1}$ -potential, then so is $q - k^2 \chi_Q$, which allows us to recover the potential using Theorem 4.3.3 together with Theorem 3.2.6. \square

For the stability estimates of the sequel we will require some continuity properties of the constant in Theorem 4.3.3 which we record as a lemma.

Lemma 4.3.6. Let \mathcal{N} be the set of x such that $\phi_x|_{\cup \partial \Omega_j}$ has a stationary point of degree greater than one. Then the constant $C_x = C(x, \cup \partial \Omega_j)$ in Theorem 4.3.3 has the following continuity properties in $\Omega \setminus \mathcal{N}$:

- i) It is continuous with respect to x .
- ii) It is continuous with respect to $\cup \partial \Omega_j$ in the C^2 norm.

Proof. Let N , Ω_j and r_1 be as in the proof of Theorem 4.3.3. Let $\Omega_j^* = \Omega_j$ for $j = 2, \dots, N$ and let

$$\Omega_1^* = \Omega_1 \setminus B_{r_1}(x).$$

The constant C_x is given by

$$\begin{aligned} C_x &= \sum_{j=1}^N C'_x \left\| \frac{\nabla \phi \cdot \mathbf{n}}{\|\nabla \phi\|^2} \right\|_{W^{1,\infty}(\partial \Omega_j^*)} + \left\| \frac{\nabla \phi \cdot \mathbf{n}}{\|\nabla \phi\|^2} \right\|_{L^\infty(\partial \Omega_j^*)} \\ &\quad \times \left(\|\phi\|_{\dot{W}^{1,\infty}(\partial \Omega_j^*)} \left\| \|\nabla \phi\|^{-2} \right\|_{L^\infty(\partial \Omega_j^*)} + \left\| \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|^2} \right\|_{L^\infty(\partial \Omega_j^*)} \right) \\ &\quad + \left(\|\phi\|_{W^{2,\infty}(\Omega_j^*)} \left\| \|\nabla \phi\|^{-2} \right\|_{W^{1,\infty}(\Omega_j^*)} + \left\| \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|^2} \right\|_{W^{1,\infty}(\Omega_j^*)} \right)^2. \end{aligned}$$

The constant C'_x appears in equation (4.8) by the use of Lemma 4.2.4 (taking $g = \phi|_{\partial \Omega_j}$ and $h = \frac{q_j \nabla \phi \cdot \mathbf{n}}{\|\nabla \phi\|^2}|_{\partial \Omega_j}$), so it is continuous with respect to x and

with respect to $\cup \partial \Omega_j$ in the C^2 norm, and as $\nabla \phi$ does not vanish inside any of the Ω_j^* , then so is C_x , concluding the proof. \square

As is to be expected, the error in the reconstruction increases the closer we move to the discontinuities of the potential, as the constant C_x blows up, and we are unable to recover at the discontinuities. It is perhaps more interesting that, for certain potentials, there are points where the reconstruction fails which are far from the discontinuities of the potential.

Indeed, consider the potential given by $q = \chi_{\Omega_1}$, where Ω_1 is the rhombus with vertices at $(0, 0)$, $(1, 1)$, $(2, 0)$ and $(1, -1)$; see Figure 4.1. Consider the problem of recovering the potential inside $\Omega = [-2, 2] \times [-2, 2]$. We might expect to be able to recover at the points $x = (-t, -t)$, for $t \in (0, 2)$, far from the potential. However, by Alessandrini's identity [2, Lemma 1], we know that the reconstructed potential \tilde{q} at x is the limit as λ tends to infinity of

$$\frac{\lambda}{\pi} \int_{\partial \Omega} e^{i\lambda \bar{\psi}_x} (\Lambda_q - \Lambda_0) [u_{\lambda, x}] = \frac{\lambda}{\pi} \int_{\Omega} e^{i\lambda \phi_x} q (1 + w_{\lambda, x}),$$

which can be rewritten as

$$\frac{\lambda}{\pi} \left(\int_{\Omega} e^{i\lambda \phi_x} q w_{\lambda, x} + \int_{\Omega \setminus \Omega_1} e^{i\lambda \phi_x} q + \int_{\Omega_1} e^{i\lambda \phi_x} q \right).$$

For the first term we can use Lemma 3.2.5 and part *ii* of Lemma 4.2.2 to obtain

$$\left| \frac{\lambda}{\pi} \int_{\Omega} e^{i\lambda \phi_x} q w_{\lambda, x} \right| \leq C \lambda^{-r} \|q\|_{D^{2,r}}^2, \quad \text{as } \lambda \rightarrow \infty,$$

for any $0 < r < 1/2$. As the potential is equal to 1 inside Ω_1 and zero in the rest of the domain, this yields

$$\tilde{q}(x) = \lim_{\lambda \rightarrow \infty} \frac{\lambda}{\pi} \int_{\Omega_1} e^{i\lambda \phi_x}.$$

Using Green's first identity twice we get

$$\begin{aligned} \frac{\lambda}{\pi} \int_{\Omega_1} e^{i\lambda \phi_x} &= \frac{1}{i\pi} \int_{\partial \Omega_1} e^{i\lambda \phi_x} \frac{\nabla \phi_x}{\|\nabla \phi_x\|^2} \cdot \mathbf{n} \\ &+ \frac{1}{\pi \lambda} \int_{\partial \Omega_1} e^{i\lambda \phi_x} \nabla \cdot \left(\frac{\nabla \phi_x}{\|\nabla \phi_x\|^2} \right) \frac{\nabla \phi_x}{\|\nabla \phi_x\|^2} \cdot \mathbf{n} \\ &- \frac{1}{\pi \lambda} \int_{\Omega_1} e^{i\lambda \phi_x} \nabla \cdot \left(\nabla \cdot \left(\frac{\nabla \phi_x}{\|\nabla \phi_x\|^2} \right) \frac{\nabla \phi_x}{\|\nabla \phi_x\|^2} \right). \end{aligned}$$

As we have seen in the proof of Theorem 4.3.3, the second and third terms converge to zero as λ tends to infinity, and for the first term we can write

$$\begin{aligned} \int_{\partial\Omega_1} e^{i\lambda\phi_x} \frac{\nabla\phi_x}{\|\nabla\phi_x\|^2} \cdot \mathbf{n} &= \int_{l_1} e^{i\lambda\phi_x} \frac{\nabla\phi_x}{\|\nabla\phi_x\|^2} \cdot \mathbf{n}_1 + \int_{l_2} e^{i\lambda\phi_x} \frac{\nabla\phi_x}{\|\nabla\phi_x\|^2} \cdot \mathbf{n}_2 \\ &\quad + \int_{l_3} e^{i\lambda\phi_x} \frac{\nabla\phi_x}{\|\nabla\phi_x\|^2} \cdot \mathbf{n}_3 + \int_{l_4} e^{i\lambda\phi_x} \frac{\nabla\phi_x}{\|\nabla\phi_x\|^2} \cdot \mathbf{n}_4 \end{aligned}$$

where

$$\begin{aligned} l_1 &= (s, s) \text{ for } s \in (0, 1), \\ l_2 &= (1 + s, 1 - s) \text{ for } s \in (0, 1), \\ l_3 &= (2 - s, -s) \text{ for } s \in (0, 1), \\ l_4 &= (1 - s, s - 1) \text{ for } s \in (0, 1). \end{aligned}$$

As we have $\phi_x(z) = (z_1 + t)^2 - (z_2 + t)^2$, we see that

$$\begin{aligned} \phi_x|_{l_1}(s) &= 0, \\ \phi_x|_{l_2}(s) &= 4s(t + 1), \\ \phi_x|_{l_3}(s) &= 4(t - s + 1), \\ \phi_x|_{l_4}(s) &= 4t(1 - s). \end{aligned}$$

Therefore, we can apply Lemma 4.2.4 to three of the sides;

$$\int_{l_j} e^{i\lambda\phi_x} \frac{\nabla\phi_x}{\|\nabla\phi_x\|^2} \cdot \mathbf{n}_j = O(\lambda^{-1/2}), \quad \text{for } j = 2, 3, 4,$$

and on the remaining side we have

$$\int_{l_1} e^{i\lambda\phi_x} \frac{\nabla\phi_x}{\|\nabla\phi_x\|^2} \cdot \mathbf{n}_1 = \int_0^1 \frac{-\sqrt{2}}{4(s+t)} ds = \frac{\sqrt{2}}{4} (\log(t) - \log(t+1)).$$

Putting everything together we obtain

$$\tilde{q}(x) = \frac{\sqrt{2}i}{4\pi} \log(1 + 1/t) \neq 0.$$

4.4 Stability

We begin with some preliminary results that we will require for the proof of the stability estimates. The operator T_g^λ is defined in (3.8).

Lemma 4.4.1. *Let $q_1, q_2 \in H^s(\mathbb{R}^2)$, where $0 < 2s \leq 1$ and let $F \in L^2(\mathbb{R}^2)$. Then there exists a constant C such that*

$$\sup_{x \in \Omega} \left| T_{w_1 w_2}^\lambda[F](x) \right| \leq C \lambda^{-2s} \|F\|_{L^2} \|q_1\|_{\dot{H}^s} \|q_2\|_{\dot{H}^s}$$

when λ is sufficiently large.

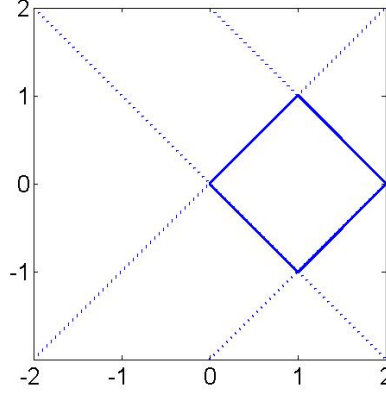


Figure 4.1: Solid lines are the discontinuities of the potential and dashed lines are points far from the discontinuities where the recovery fails.

Proof. By the Hölder and Hardy–Littlewood–Sobolev inequalities,

$$\begin{aligned} \left| T_{w_1 w_2}^\lambda [F](x) \right| &\leq \lambda \|F w_1 w_2\|_{L^1} \\ &\leq \lambda \|F\|_{L^2} \|w_1\|_{L^4} \|w_2\|_{L^4} \\ &\leq C \lambda \|F\|_{L^2} \|w_1\|_{\dot{H}^{1/2}} \|w_2\|_{\dot{H}^{1/2}}. \end{aligned}$$

As $(I - S_q^\lambda)^{-1}$ is bounded for large λ (see Lemma 3.2.3), this yields

$$\left| T_{w_1 w_2}^\lambda [F](x) \right| \leq C \lambda \|F\|_{L^2} \left\| S_1^\lambda [q_1] \right\|_{\dot{H}^{1/2}} \left\| S_1^\lambda [q_2] \right\|_{\dot{H}^{1/2}}.$$

Applying Lemma 3.2.2 twice concludes the proof. \square

We will also require the following crude bound for Bukhgeim solutions.

Lemma 4.4.2. *Let $0 < r < 1/2$, and let q be a piecewise $W^{2,1}$ -potential defined on a bounded planar domain Ω with diameter d . Then there exists a constant C depending on Ω such that the Bukhgeim solutions satisfy*

$$\|u_{\lambda,x}\|_{H^1(\Omega)} \leq C e^{\lambda d^2} \left(1 + \|q\|_{D^{2,r}}^2 \right)$$

whenever λ is sufficiently large.

Proof. Writing $u = u_{\lambda,x}$,

$$\begin{aligned} \|u\|_{H^1(\Omega)} &\leq C \left(\|u\|_{L^2(\Omega)} + \|\partial_{\bar{z}} u\|_{L^2(\Omega)} + \|\partial_z u\|_{L^2(\Omega)} \right) \\ &= C \left\| e^{i\lambda\psi} (1+w) \right\|_{L^2(\Omega)} + C \left\| \partial_z^{-1} q e^{i\lambda\psi} (1+w) \right\|_{L^2(\Omega)} \\ &\quad + C \left\| \partial_{\bar{z}}^{-1} q e^{i\lambda\psi} (1+w) \right\|_{L^2(\Omega)}. \end{aligned}$$

Note that ∂_z^{-1} and $\partial_{\bar{z}}^{-1}$ are bounded operators, as q has compact support (see for example [8, Theorem 4.3.12]). As we have $\|\cdot\|_{L^2(\Omega)} \leq C \|\cdot\|_{L^4(\Omega)}$, then using Hölder's inequality leads to

$$\begin{aligned} \|u\|_{H^1(\Omega)} &\leq C \left\| e^{i\lambda\psi} \right\|_{L^\infty(\Omega)} (1 + \|q\|_{L^4}) \|1 + w\|_{L^4(\Omega)} \\ &\leq C e^{\lambda d^2} (1 + \|q\|_{L^4}) (1 + \|w\|_{L^4(\Omega)}), \end{aligned}$$

and by the Hardy-Littlewood-Sobolev inequality we get

$$\|u\|_{H^1(\Omega)} \leq C e^{\lambda d^2} (1 + \|q\|_{L^4}) (1 + \|w\|_{\dot{H}^{1/2}}).$$

As $(I - S_q^\lambda)^{-1}$ is bounded for λ sufficiently large (Lemma 3.2.3)

$$\|u\|_{H^1(\Omega)} \leq C e^{\lambda d^2} (1 + \|q\|_{L^4}) \left(1 + \left\| S_1^\lambda[q] \right\|_{\dot{H}^{1/2}} \right).$$

Now, by Lemma 3.2.2 we get

$$\|u\|_{H^1(\Omega)} \leq C e^{\lambda d^2} (1 + \|q\|_{L^4}) (1 + \|q\|_{\dot{H}^r}).$$

Using Sobolev embedding (see for example [1, Theorem 4.12]) we have

$$\|q\|_{L^4} \leq C \sum_{j=1}^N \|q_j\|_{L^4} \leq \|q\|_{D^{2,r}}.$$

Using part *ii* of Lemma 4.2.2 Lemma concludes the proof. \square

We now prove a conditional stability estimate for reconstruction from the DtN map in the L^∞ norm. Note that the result is interesting when there is a priori knowledge of where, approximately, the discontinuities lie, as the constant term depends on the point under consideration with respect to the discontinuities. The result has been stated in the following form as in practical situations one could consider where a potential might lie, given a noisy reconstruction of it. If the assumption that the discontinuities of the potentials are close to each other was dropped, the constant C_x would depend on both $\{\partial\Omega_{1,j}\}$ and $\{\partial\Omega_{2,j}\}$, the discontinuities of q_1 and q_2 respectively.

Theorem 4.4.3. *Let $0 < s - 2 < 2r < 1$ and let q_1, q_2 be piecewise $W^{s,1}$ -potentials supported on a bounded Lipschitz domain Ω in \mathbb{R}^2 such that their discontinuities are close enough with respect to the C^2 norm. Then, for almost every $x \in \Omega$, there exists a constant $C_x = C(x, \cup \partial\Omega_{1,j})$ such that*

$$|q_1(x) - q_2(x)| \leq C_x \left| \ln \|\Lambda_{q_1} - \Lambda_{q_2}\| \right|^{1-s/2} (\kappa + \kappa^3)$$

whenever Λ_{q_1} and Λ_{q_2} are close enough, where $\kappa = \max\{\|q_1\|_{D^{s,r}}, \|q_2\|_{D^{s,r}}\}$.

Proof. Let d be the diameter of Ω and let

$$\lambda = -\frac{1}{6d^2} \ln \|\Lambda_{q_1} - \Lambda_{q_2}\|. \quad (4.16)$$

Note that whenever Λ_{q_1} and Λ_{q_2} are sufficiently close, then λ can be as large as we need.

By the triangle inequality,

$$\begin{aligned} |q_1(x) - q_2(x)| &\leq \left| q_1(x) - T_{1+w_1}^\lambda[q_1](x) \right| + \left| q_2(x) - T_{1+w_2}^\lambda[q_2](x) \right| \\ &\quad + \left| T_{1+w_1}^\lambda[q_1](x) - T_{1+w_2}^\lambda[q_2](x) \right|. \end{aligned} \quad (4.17)$$

We can use Theorem 4.3.3 on the first two terms to obtain

$$\left| q_j(x) - T_{1+w_j}^\lambda[q_j](x) \right| \leq C_x \lambda^{1-s/2} (\kappa + \kappa^2) \quad (4.18)$$

where $C_x = C(x, \cup \partial\Omega_{1,j})$. We can take the same constant C_x for both terms as it is continuous with respect to the discontinuities in the C^2 norm (due to Lemma 4.3.6). For the last term we have

$$\begin{aligned} \left| T_{1+w_1}^\lambda[q_1] - T_{1+w_2}^\lambda[q_2] \right| &\leq \left| T_{(1+w_1)(1+w_2)}^\lambda[q_1 - q_2] \right| + \left| T_{w_1}^\lambda[q_2] \right| \\ &\quad + \left| T_{w_2}^\lambda[q_1] \right| + \left| T_{w_1 w_2}^\lambda[q_1 - q_2] \right|. \end{aligned}$$

By Lemma 3.2.5 and part *ii* of Lemma 4.2.2 we obtain

$$\left\| T_{w_2}^\lambda[q_1] \right\|_{L^\infty} \leq C \lambda^{1-s/2} \kappa^2 \quad \text{and} \quad \left\| T_{w_1}^\lambda[q_2] \right\|_{L^\infty} \leq C \lambda^{1-s/2} \kappa^2 \quad (4.19)$$

and by Lemma 4.4.1 and part *ii*) of Lemma 4.2.2 we obtain

$$\left\| T_{w_1 w_2}^\lambda[q_1 - q_2] \right\|_{L^\infty} \leq C \lambda^{1-s/2} \kappa^3. \quad (4.20)$$

Let u_j be Bukhgeim solutions to $\Delta u_j = q_j u_j$. Then we have

$$\begin{aligned} \left\| T_{(1+w_1)(1+w_2)}^\lambda[q_1 - q_2] \right\|_{L^\infty} &= \frac{\lambda}{\pi} \left\| \int_{\Omega} (q_1 - q_2) u_1 u_2 \right\|_{L^\infty} \\ &= \frac{\lambda}{\pi} \left\| \int_{\partial\Omega} f_1 \Lambda_{q_2}[f_2] - f_2 \Lambda_{q_1}[f_1] \right\|_{L^\infty}, \end{aligned}$$

where $f_j = u_j|_{\partial\Omega}$. As the DtN is a self-adjoint operator, we have that

$$\frac{\lambda}{\pi} \left\| \int_{\partial\Omega} f_1 \Lambda_{q_2}[f_2] - f_2 \Lambda_{q_1}[f_1] \right\|_{L^\infty} = \frac{\lambda}{\pi} \left\| \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2}) [f_1] f_2 \right\|_{L^\infty}.$$

For $f \in H^{1/2}(\partial\Omega)$ the DtN map satisfies $\Lambda_q[f] \in H^{-1/2}(\partial\Omega)$, where the space $H^{-1/2}(\partial\Omega)$ is the dual of $H^{1/2}(\partial\Omega)$. Thus, for any $x \in \Omega$, we have

$$\begin{aligned} \frac{\lambda}{\pi} \left| \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2}) [f_1] f_2 \right| &\leq \lambda \|\Lambda_{q_1} - \Lambda_{q_2}\| \|f_1\|_{H^{1/2}(\partial\Omega)} \|f_2\|_{H^{1/2}(\partial\Omega)} \\ &\leq \lambda \|\Lambda_{q_1} - \Lambda_{q_2}\| \|u_1\|_{H^1(\Omega)} \|u_2\|_{H^1(\Omega)} \end{aligned}$$

and we can use Lemma 4.4.2 to obtain

$$\frac{\lambda}{\pi} \left| \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2}) [f_1] f_2 \right| \leq C \lambda e^{2\lambda d^2} \|\Lambda_{q_1} - \Lambda_{q_2}\| (1 + \kappa^4). \quad (4.21)$$

Inserting (4.18), (4.19), (4.20) and (4.21) into (4.17), and noting that we have $\lambda < e^{\lambda d}$ for large λ , leads to

$$|q_1(x) - q_2(x)| \leq C_x \lambda^{1-s/2} (\kappa + \kappa^3) + C \|\Lambda_{q_1} - \Lambda_{q_2}\| e^{3\lambda d^2} (1 + \kappa^4).$$

Taking λ as in (4.16) we obtain

$$\begin{aligned} |q_1(x) - q_2(x)| &\leq C_x \left(\frac{-1}{6d^2} \ln \|\Lambda_{q_1} - \Lambda_{q_2}\| \right)^{1-s/2} (1 + \kappa^3) \\ &\quad + C \|\Lambda_{q_1} - \Lambda_{q_2}\|^{1/2} (1 + \kappa^4) \end{aligned}$$

where the second term can be omitted for $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ small enough, concluding the proof. \square

To obtain the stability estimate from the scattering amplitude we adapt the proof of Stefanov (see [59]) to the two-dimensional case. Due to the severe ill-posedness of the problem, a norm for the scattering amplitude which penalizes the higher components of the frequency spectrum is required. Let q be a potential supported on the unit disk, then we define the norm for its scattering amplitude at a fixed energy k^2 as

$$\|A_q\|_k^2 = \sum_{n,m \in \mathbb{Z}} \left(\frac{3+3|n|}{k} \right)^{2|n|} \left(\frac{3+3|m|}{k} \right)^{2|m|} |a_q^{(n,m)}|^2,$$

where $a_q^{(n,m)}$ are the Fourier coefficients of A_q

$$A_q(\eta, \theta) = \sum_{n,m \in \mathbb{Z}} a_q^{(n,m)} e^{in\eta + im\theta}.$$

Before passing to the proof, we recall the definition of the single layer potential operator

$$\mathcal{S}_q[f](x) = \int_{\partial\Omega} G_q(x, y) f(y) dy,$$

where G_q is the Green's function that satisfies

$$(-\Delta + q - k^2) G_q(x, y) = \delta(x - y), \quad \forall x, y \in \mathbb{R}^2,$$

and the outgoing Sommerfeld radiation condition given in equation (3.2).

Lemma 4.4.4. *Let $q_1, q_2 \in L^\infty(\mathbb{R}^2)$ be two real-valued potentials supported in the unit disk. Then there exists a constant $C_k = C(k)$ such that*

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\mathbb{S}^1) \rightarrow H^{-1/2}(\mathbb{S}^1)} \leq C_k \|A_{q_1} - A_{q_2}\|_k.$$

Proof. Using Nachman's formula [50, Theorem 1.6] we have

$$\begin{aligned} \Lambda_{q_1-k^2} - \Lambda_{q_2-k^2} &= \mathcal{S}_{q_1}^{-1} - \mathcal{S}_{q_2}^{-1} \\ &= \mathcal{S}_{q_1}^{-1} (\mathcal{S}_{q_2} - \mathcal{S}_{q_1}) \mathcal{S}_{q_2}^{-1}. \end{aligned}$$

As \mathcal{S}_q is a bounded and invertible mapping from $H^{-1/2}(\mathbb{S}^1)$ to $H^{1/2}(\mathbb{S}^1)$ (see [39, Proposition A.1]), we have

$$\|\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2}\|_{H^{1/2}(\mathbb{S}^1) \rightarrow H^{-1/2}(\mathbb{S}^1)} \leq C \|\mathcal{S}_{q_1} - \mathcal{S}_{q_2}\|_{H^{-1/2}(\mathbb{S}^1) \rightarrow H^{1/2}(\mathbb{S}^1)}.$$

Letting $B_x = (1 - \Delta_x)^{1/4}$ we write

$$\begin{aligned} &\|\mathcal{S}_{q_1} - \mathcal{S}_{q_2}\|_{H^{-1/2}(\mathbb{S}^1) \rightarrow H^{1/2}(\mathbb{S}^1)} \\ &= \sup_{\|f\|=1} \left\| \int_{\mathbb{S}^1} (G_{q_1}(x, y) - G_{q_2}(x, y)) f(y) dy \right\|_{H^{1/2}(\mathbb{S}^1)} \\ &= \sup_{\|f\|=1} \left(\int_{\mathbb{S}^1} \left(B_x \int_{\mathbb{S}^1} (G_{q_1}(x, y) - G_{q_2}(x, y)) f(y) dy \right)^2 dx \right)^{1/2} \\ &= \sup_{\|f\|=1} \left(\int_{\mathbb{S}^1} \left(\int_{\mathbb{S}^1} B_x (G_{q_1}(x, y) - G_{q_2}(x, y)) f(y) dy \right)^2 dx \right)^{1/2} \end{aligned}$$

by Parseval's identity we have

$$= \sup_{\|f\|=1} \left(\int_{\mathbb{S}^1} \left(\int_{\mathbb{S}^1} (B_y B_x (G_{q_1}(x, y) - G_{q_2}(x, y))) B_y^{-1} f(y) dy \right)^2 dx \right)^{1/2}$$

using Minkowski's integral inequality we get

$$\begin{aligned} &\leq \sup_{\|f\|=1} \int_{\mathbb{S}^1} \left(\int_{\mathbb{S}^1} ((B_y B_x (G_{q_1}(x, y) - G_{q_2}(x, y))) B_y^{-1} f(y))^2 dx \right)^{1/2} dy \\ &= \sup_{\|f\|=1} \int_{\mathbb{S}^1} |B_y^{-1} f(y)| \left(\int_{\mathbb{S}^1} (B_y B_x (G_{q_1}(x, y) - G_{q_2}(x, y)))^2 dx \right)^{1/2} dy \end{aligned}$$

and using the Cauchy-Schwarz inequality

$$\begin{aligned} &\leq \sup_{\|f\|=1} \|f\|_{H^{-1/2}(\mathbb{S}^1)} \|G_{q_1} - G_{q_2}\|_{H^{1/2}(\mathbb{S}^1) \otimes H^{1/2}(\mathbb{S}^1)} \\ &= \|G_{q_1} - G_{q_2}\|_{H^{1/2}(\mathbb{S}^1) \otimes H^{1/2}(\mathbb{S}^1)}. \end{aligned}$$

From [6, Theorem 2.2] we know that

$$\begin{aligned} G_{q_1}(x, y) - G_{q_2}(x, y) &= \sum_{n, m \in \mathbb{Z}} \frac{(-1)^n}{16} i^{n+m} \left(a_{q_1}^{(n, m)} - a_{q_2}^{(n, m)} \right) \\ &\quad \times H_n^{(1)}(k|x|) H_m^{(1)}(k|y|) e^{in\phi_x + im\phi_y} \end{aligned}$$

where $H^{(1)}$ denotes the Hankel function of the first kind. Now, using Parseval's identity and the bound for the Hankel function in [6, Lemma 2.3], there exists $C'_k = C(k)$ such that

$$\begin{aligned} &\|G_{q_1} - G_{q_2}\|_{H^{1/2}(\mathbb{S}^1) \otimes H^{1/2}(\mathbb{S}^1)}^2 \\ &\leq \sum_{n, m \in \mathbb{Z}} (1+n^2)^{1/2} (1+m^2)^{1/2} \left| a_{q_1}^{(n, m)} - a_{q_2}^{(n, m)} \right|^2 \left| H_n^{(1)}(k) \right|^2 \left| H_m^{(1)}(k) \right|^2 \\ &\leq C'_k \sum_{n, m \in \mathbb{Z}} (1+n^2)^{1/2} (1+m^2)^{1/2} \left| a_{q_1}^{(n, m)} - a_{q_2}^{(n, m)} \right|^2 |n|!^2 |m|!^2 \left(\frac{3}{k} \right)^{2|n|+2|m|} \\ &\leq C'_k \sum_{n, m \in \mathbb{Z}} \left(\frac{3+3|n|}{k} \right)^{2|n|} \left(\frac{3+3|m|}{k} \right)^{2|m|} \left| a_{q_1}^{(n, m)} - a_{q_2}^{(n, m)} \right|^2 \end{aligned}$$

concluding the proof. \square

Corollary 4.4.5. *Let $0 < s-2 < 2r < 1$ and let q_1, q_2 be two piecewise- $W^{s,1}$ real-valued potentials supported on a bounded domain in \mathbb{R}^2 such that their discontinuities are close enough in the C^2 norm. Then, for almost every $x \in \Omega$, there exist constants $C_x = C(x, \cup \partial\Omega_j)$, $C_k = C(k)$, such that*

$$|q_1(x) - q_2(x)| \leq C_x \left| \ln \left(C_k \|A_{q_1} - A_{q_2}\|_k \right) \right|^{1-s/2} (\kappa + \kappa^3)$$

whenever A_{q_1} and A_{q_2} are close enough, where $\kappa = \max\{\|q_1\|_{D^{s,r}}, \|q_2\|_{D^{s,r}}\}$.

Chapter 5

Averaging procedures for potential reconstruction

5.1 Introduction

In this chapter we will exploit the fact that the reconstruction formula in the Bukhgeim approach commutes, in some sense, with taking averages. This allows us to obtain a reconstruction formula for complex-valued potentials in $H^s(\mathbb{R}^2)$ for any $s > 0$, and also to obtain reconstruction formulas with improved convergence (see Chapter 7) compared to the standard formula.

Let φ be a Schwartz function, supported in the unit ball, that satisfies

$$\varphi(x) \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \varphi(x) dx = 1,$$

and we write $\varphi_\sigma(x) := \sigma^{-2} \varphi(\sigma^{-1}x)$ for $\sigma > 0$. For $x \in \Omega$ we denote the boundary information at frequency λ by

$$BI_\lambda(x) := \frac{\lambda}{\pi} \left\langle (\Lambda_q - \Lambda_0)[u_{\lambda,x}|_{\partial\Omega}], e^{i\lambda\bar{\psi}_x}|_{\partial\Omega} \right\rangle$$

where $u_{\lambda,x}$ are Bukhgeim solutions of the form (3.5). The following result will be proved in Section 5.3.

Theorem 5.1.1. *Let $s > 0$, let $q \in H^s(\mathbb{R}^2)$ be a complex-valued potential supported in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and let $\sigma = \lambda^{-1/4}$. Then*

$$\lim_{\lambda \rightarrow \infty} \varphi_\sigma * BI_\lambda(x) = q(x) \quad \text{a.e. } x \in \Omega.$$

Together with the proof of Theorem 5.1.1, we will also give a bound for the dimension of the set of points where the recovery fails and provide an estimate for the decay rate of the error of the reconstruction if the potential satisfies some stronger hypothesis.

In Section 5.4 we present a different reconstruction formula which also relies on an averaging procedure. In this case the averaging is taken over rotations of the potential, and over a range of the frequencies λ . We are able to prove a result similar to Theorem 5.1.1 for this formula, but requiring some additional local regularity. This formula is of interest for the purpose of the numerical implementation of a reconstruction algorithm (see Chapter 7).

5.2 Preliminaries

We introduce two lemmas for controlling the norm and the rate of convergence of the mollified potential.

Lemma 5.2.1. *Let $q \in H^s(\mathbb{R}^2)$ with $0 < s < s'$. Then*

$$\|\varphi_\sigma * q\|_{\dot{H}^{s'}} \leq C \sigma^{s-s'} \|q\|_{\dot{H}^s}.$$

Proof. Given that $\widehat{\varphi_\sigma * q}(\xi) = \widehat{\varphi}(\sigma\xi)$, by Hölder's inequality we have

$$\begin{aligned} \|\varphi_\sigma * q\|_{\dot{H}^{s'}} &\leq \|q\|_{\dot{H}^s} \sup_{\xi} \left\{ |\xi|^{s'-s} \widehat{\varphi}(\sigma\xi) \right\} \\ &= \sigma^{s-s'} \|q\|_{\dot{H}^s} \sup_{\zeta} \left\{ |\zeta|^{s'-s} \widehat{\varphi}(\zeta) \right\}. \end{aligned}$$

As φ belongs to the Schwartz space, the proof is concluded. \square

Lemma 5.2.2. *Let $q \in C^{0,\alpha}(\mathbb{R}^2)$. Then*

$$\|\varphi_\sigma * q - q\|_{L^\infty} \leq \sigma^\alpha |q|_{C^{0,\alpha}}.$$

Proof. For $x \in \mathbb{R}^2$ we have

$$\begin{aligned} |\varphi_\sigma * q(x) - q(x)| &= \left| q(x) - \int_{B(x,\sigma)} q(y) \varphi_\sigma(x-y) dy \right| \\ &\leq \int_{B(x,\sigma)} |q(x) - q(y)| \varphi_\sigma(x-y) dy. \end{aligned}$$

Then by Hölder's inequality,

$$\begin{aligned} |\varphi_\sigma * q(x) - q(x)| &\leq \int_{B(x,\sigma)} \frac{|q(x) - q(y)|}{|x-y|^\alpha} |x-y|^\alpha \varphi_\sigma(x-y) dy \\ &\leq |q|_{C^{0,\alpha}} \int_{B(x,\sigma)} |x-y|^\alpha \varphi_\sigma(x-y) dy \\ &\leq \sigma^\alpha |q|_{C^{0,\alpha}}, \end{aligned}$$

which completes the proof. \square

5.3 Proof of Theorem 5.1.1

We use $\dim_H\{E\}$ to denote the Hausdorff dimension of E . Theorem 5.1.1 is a consequence of the following more precise statement.

Theorem 5.3.1. *Let $s > 0$, let $q \in H^s(\mathbb{R}^2)$ be a complex-valued potential supported in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and let $\sigma = \lambda^{-1/4}$. Then*

$$\dim_H \left\{ x : \lim_{\lambda \rightarrow \infty} \varphi_\sigma * BI_\lambda(x) \neq q(x) \right\} \leq 2 - 2s.$$

Proof. Let T_g^λ as in (3.8). By Alessandrini's identity (2.2) we have

$$BI_\lambda(x) = \frac{\lambda}{\pi} \int_{\Omega} e^{i\lambda\phi_x(z)} q(z)(1 + w_{\lambda,x}(z)) dz = T^\lambda[q](x) + T_w^\lambda[q](x)$$

leading to

$$\varphi_\sigma * BI_\lambda = \varphi_\sigma * T^\lambda[q] + \varphi_\sigma * T_w^\lambda[q].$$

Let $q_\sigma := \varphi_\sigma * q$. Using the triangle inequality we have

$$|\varphi_\sigma * BI_\lambda - q| \leq |\varphi_\sigma * T^\lambda[q] - q_\sigma| + |q_\sigma - q| + |\varphi_\sigma * T_w^\lambda[q]|. \quad (5.1)$$

For the first term, we can use Fubini's theorem to obtain

$$\begin{aligned} \varphi_\sigma * T^\lambda[q](x) &= \frac{\lambda}{\pi} \int_{\mathbb{R}^2} \varphi_\sigma(y) \int_{\Omega} e^{i\lambda\phi_{x-y}(z)} q(z) dz dy \\ &= \frac{\lambda}{\pi} \int_{\mathbb{R}^2} \varphi_\sigma(y) \int_{\Omega} e^{i\lambda\phi_x(z+y)} q(z) dz dy \\ &= \frac{\lambda}{\pi} \int_{\mathbb{R}^2} e^{i\lambda\phi_x(z)} \int_{\mathbb{R}^2} \varphi_\sigma(y) q(z-y) dy dz \end{aligned}$$

leading to

$$\varphi_\sigma * T^\lambda[q] = T^\lambda[q_\sigma].$$

Using Lemma 3.2.4 and Lemma 5.2.1, for s' satisfying $1 < s' < 3$, we have

$$\left| T^\lambda[q_\sigma] - q_\sigma \right| \leq C \lambda^{\frac{1-s'}{2}} \|q_\sigma\|_{\dot{H}^{s'}} \leq C \lambda^{\frac{1-s'}{2}} \sigma^{s-s'} \|q\|_{\dot{H}^s}.$$

To deal with the third term, we note that by Lemma 3.2.5 we have that

$$|\varphi_\sigma * T_w^\lambda[q]| \leq \|T_w^\lambda[q]\|_{L^\infty} \leq C \lambda^{-s} \|q\|_{\dot{H}^s}^2.$$

Plugging these estimates into (5.1), we can see that

$$|\varphi_\sigma * BI_\lambda - q| \leq C \lambda^{\frac{1-s'}{2}} \sigma^{s-s'} \|q\|_{\dot{H}^s} + C \lambda^{-s} \|q\|_{\dot{H}^s}^2 + |q_\sigma - q|. \quad (5.2)$$

Thus we have

$$|\varphi_\sigma * BI_\lambda(x) - q(x)| \leq C \lambda^{-\kappa} (\|q\|_{\dot{H}^s} + \|q\|_{\dot{H}^s}^2) + |q_\sigma(x) - q(x)|$$

for any κ satisfying $0 < \kappa < \min\{(1+s)/4, s\}$. It only remains to see that

$$\dim_H \left\{ x : \lim_{\sigma \rightarrow 0} q_\sigma(x) \neq q(x) \right\} \leq 2 - 2s$$

which follows from, for example, [13, Lemma A.1]. \square

Remark 5.3.2. *If the potential satisfies the Hölder condition in some neighborhood of the point to reconstruct, we can obtain an estimation for the decay rate of the error term. More precisely, for x such that $|q|_{C^{0,\alpha}(B(x,r))} < \infty$ for some $r, \alpha > 0$, we have*

$$|\varphi_\sigma * BI_\lambda(x) - q(x)| \leq C \lambda^{-\min\{s,\alpha\}/4} \left(\|q\|_{\dot{H}^s} + \|q\|_{\dot{H}^s}^2 + |q|_{C^{0,\alpha}(B(x,r))} \right).$$

This follows from using Lemma 5.2.2 to bound $|q_\sigma(x) - q(x)|$ in (5.2).

Corollary 5.3.3. *Let $s > 0$ and let $q \in H^s(\mathbb{R}^2)$ be a real-valued potential with compact support. Then the potential can be recovered from the scattering amplitude almost everywhere.*

This follows from the expressions in Section 3.1 to compute Λ_{q-k^2} from the scattering amplitude at energy k^2 .

5.4 Polar averaging

First we introduce some averaging operators which are used to express our recovery formula involving *polar averaging*.

For $f : \mathbb{R}^+ \times [0, 2\pi) \rightarrow \mathbb{C}$ we define the *angular averaging* operator by

$$A_{\text{ang}}[f](r, \theta) := \frac{1}{2\pi} \int_0^{2\pi} f(r, \alpha) d\alpha,$$

and the *radial smoothing* operator by

$$S_{\text{rad}}[f](r, \theta) := \int_0^1 f(r(1+s)^{-1/2}, \theta) ds.$$

For $\lambda \in \mathbb{R}^+$ and $F \in L_{\text{loc}}^1(\mathbb{R}^+)$ we define the *frequency averaging* operator

$$A_{\text{freq}}[F](\lambda) := \frac{1}{\lambda} \int_\lambda^{2\lambda} F(t) dt.$$

In the following lemma, we connect the frequency averaging with the radial smoothing operator.

Lemma 5.4.1. *Let $f \in L^1(\mathbb{R}^2)$. Then*

$$A_{\text{freq}}[T^{(\cdot)}[f](0)](\lambda) = T^\lambda[S_{\text{rad}}[f]](0).$$

Proof. First we note that by a change of variables

$$A_{\text{freq}}[F](\lambda) = \frac{1}{\lambda} \int_{\lambda}^{2\lambda} F(t) dt = \int_0^1 F(\lambda(1+s)) ds$$

so that $A_{\text{freq}}[T^{(\cdot)}[f](0)](\lambda)$ can be written as

$$\int_0^1 T^{\lambda(1+s)}[f](0) ds = \int_0^1 \frac{\lambda(1+s)}{\pi} \int_{\mathbb{R}^2} e^{i\lambda(1+s)(z_1^2 - z_2^2)} f(z) dz ds.$$

Now, switching to polar coordinates, changing variables $r = (1+s)^{1/2} \rho$ and applying Fubini's theorem, we find

$$\begin{aligned} & \int_0^1 \frac{\lambda(1+s)}{\pi} \int_{\mathbb{R}^2} e^{i\lambda(1+s)(z_1^2 - z_2^2)} f(z) dz ds \\ &= \int_0^1 \frac{\lambda(1+s)}{\pi} \int_0^\infty \int_0^{2\pi} e^{i\lambda(1+s)\rho^2 \cos(2\theta)} f(\rho, \theta) d\theta \rho d\rho ds \\ &= \frac{\lambda}{\pi} \int_0^1 \int_0^\infty \int_0^{2\pi} f\left(r(1+s)^{-1/2}, \theta\right) e^{i\lambda r^2 \cos(2\theta)} d\theta r dr ds \\ &= \frac{\lambda}{\pi} \int_0^\infty \int_0^{2\pi} \int_0^1 f\left(r(1+s)^{-1/2}, \theta\right) ds e^{i\lambda r^2 \cos(2\theta)} d\theta r dr \\ &= \frac{\lambda}{\pi} \int_0^\infty \int_0^{2\pi} S_{\text{rad}}[f](r, \theta) e^{i\lambda r^2 \cos(2\theta)} d\theta r dr \\ &= T^\lambda[S_{\text{rad}}[f]](0) \end{aligned}$$

and the proof is concluded. \square

We use the notation C_0^k with $k \in \mathbb{N}$ to refer to the space of continuous functions with k continuous derivatives and compact support. The following lemma is useful to see that the angular averaging operator preserves regularity.

Lemma 5.4.2. *If $f \in C_0^k(\mathbb{R}^2)$, then $A_{\text{ang}}[f] \in C_0^k(\mathbb{R}^2)$.*

Proof. For the case $k = 0$, note that f is uniformly continuous given that it has compact support. Thus, for any $\epsilon > 0$ there exists some $\delta > 0$ such that, for $x, y \in \mathbb{R}^2$ satisfying $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$, so that

$$|A_{\text{ang}}[f](x) - A_{\text{ang}}[f](y)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(|x|, \theta) - f(|y|, \theta) d\theta \right| < \epsilon$$

whenever $|x - y| < \delta$.

For $k > 0$, as $A_{\text{ang}}[f]$ is radial, the angular derivatives are zero. For the radial derivative, we note that

$$\frac{\partial^k}{\partial r^k} A_{\text{ang}}[f](r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^k f}{\partial r^k}(r, \alpha) d\alpha.$$

Applying the previous argument to the derivatives of the function concludes the proof. \square

The following lemma describes how S_{rad} , considered here as acting on one-dimensional functions,

$$S_{\text{rad}}[f](r) := \int_0^1 f(r(1+s)^{-1/2}) ds,$$

regularizes away from the origin while at the same time it preserves regularity in the whole domain.

Lemma 5.4.3. *Let $f \in L^1(\mathbb{R}^+)$. Then*

- (i) $S_{\text{rad}}[f] \in C^0(0, \infty)$.
- (ii) *If $f \in C^k(0, \infty)$, then $S_{\text{rad}}[f] \in C^{k+1}(0, \infty)$.*
- (iii) *If $f \in C^k[0, \infty)$, then $S_{\text{rad}}[f] \in C^k[0, \infty)$.*
- (iv) *If $\text{supp}(f) \subset (a, b)$, then $\text{supp}(S_{\text{rad}}[f]) \subset (a, b\sqrt{2})$.*

Proof. Let $\epsilon > 0$ and write $g = S_{\text{rad}}[f]$. Then

$$g(t+\epsilon) - g(t) = \int_0^1 f((t+\epsilon)(1+s)^{-1/2}) ds - \int_0^1 f(t(1+s)^{-1/2}) ds.$$

Changing variables in the first integral by taking

$$s' = \frac{t+\epsilon}{t}(1+s)^{-1/2} \tag{5.3}$$

and in the second integral taking

$$s' = (1+s)^{-1/2} \tag{5.4}$$

we have

$$\begin{aligned} g(t+\epsilon) - g(t) &= 2 \int_{\frac{1}{\sqrt{2}} + \frac{\epsilon}{t\sqrt{2}}}^{1+\frac{\epsilon}{t}} f(ts) \left(\frac{t+\epsilon}{t}\right)^2 s^{-3} ds - 2 \int_{\frac{1}{\sqrt{2}}}^1 f(ts) s^{-3} ds \\ &= 2 \left(\frac{t+\epsilon}{t}\right)^2 \int_1^{1+\frac{\epsilon}{t}} f(ts) s^{-3} ds - 2 \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}} + \frac{\epsilon}{t\sqrt{2}}} f(ts) s^{-3} ds \\ &\quad + 2 \frac{2t\epsilon + \epsilon^2}{t^2} \int_{\frac{1}{\sqrt{2}} + \frac{\epsilon}{t\sqrt{2}}}^1 f(ts) s^{-3} ds. \end{aligned}$$

Given some fixed $t > 0$ we have $f(t s) s^{-3} \in L^1((1/\sqrt{2}, \infty))$. By absolute continuity with respect to the Lebesgue measure, the first two integrals are smaller than δ for any $\delta > 0$ by taking ϵ small enough. The third term is smaller than $6 \epsilon t^{-1} \|f(t s) s^{-3}\|_{L^1((1/\sqrt{2}, \infty))}$. Therefore we know that g is right-continuous in $(0, \infty)$. Taking $\epsilon < t$ a similar change of variables yields

$$\begin{aligned} g(t) - g(t - \epsilon) &= 2 \int_{1-\frac{\epsilon}{t}}^1 f(t s) s^{-3} ds - 2 \left(\frac{t - \epsilon}{t} \right)^2 \int_{\frac{1}{\sqrt{2}} - \frac{\epsilon}{t\sqrt{2}}}^{\frac{1}{\sqrt{2}}} f(t s) s^{-3} ds \\ &\quad + 2 \left(\frac{2t\epsilon - \epsilon^2}{t^2} \right) \int_{\frac{1}{\sqrt{2}}}^{1-\frac{\epsilon}{t}} f(t s) s^{-3} ds \end{aligned}$$

and by the same arguments we see that g is left-continuous in $(0, \infty)$, concluding the proof of (i).

Now, differentiating under the sign of the integral we obtain

$$g^{(k)}(t) = \int_0^1 (1+s)^{-k/2} f^{(k)}(t(1+s)^{-1/2}) ds$$

and using the same change of variables as in (5.3) and (5.4) we obtain

$$\begin{aligned} &g^{(k)}(t + \epsilon) - g^{(k)}(t) \\ &= \frac{2(t + \epsilon)^{2-k}}{t^{2-k}} \int_{\frac{1}{\sqrt{2}} + \frac{\epsilon}{t\sqrt{2}}}^{1+\frac{\epsilon}{t}} f^{(k)}(t s) s^{k-3} ds - 2 \int_{\frac{1}{\sqrt{2}}}^1 f^{(k)}(t s) s^{k-3} ds \\ &= \frac{2(t + \epsilon)^{2-k}}{t^{2-k}} \int_1^{1+\frac{\epsilon}{t}} f^{(k)}(t s) s^{k-3} ds - 2 \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}} + \frac{\epsilon}{t\sqrt{2}}} f^{(k)}(t s) s^{k-3} ds \\ &\quad + 2 \frac{(t + \epsilon)^{2-k} - t^{2-k}}{t^{2-k}} \int_{\frac{1}{\sqrt{2}} + \frac{\epsilon}{t\sqrt{2}}}^1 f^{(k)}(t s) s^{k-3} ds. \end{aligned}$$

For $t > 0$ we can divide by ϵ and take the limit as ϵ tends to 0 to get

$$g^{(k+1)}(t) = \frac{2}{t} f^{(k)}(t) - \frac{2^{2-k/2}}{t} f^{(k)}(t/\sqrt{2}) + \frac{4-2k}{t} \int_{\frac{1}{\sqrt{2}}}^1 f^{(k)}(t s) s^{k-3} ds.$$

As the right-hand side is continuous in $(0, \infty)$, so is $g^{(k+1)}$, and (ii) is proved.

To prove (iii) just see that for any $\delta > 0$ we can take $\epsilon > 0$ such that for $t < \epsilon$ then $|f^{(k)}(t) - f^{(k)}(0)| < \delta$. Thus

$$\begin{aligned} |g^{(k)}(\epsilon) - g^{(k)}(0)| &\leq \int_0^1 (1+s)^{-k/2} |f^{(k)}(\epsilon(1+s)^{-1/2}) - f^{(k)}(0)| ds \\ &< \int_0^1 (1+s)^{-k/2} \delta ds \leq \delta. \end{aligned}$$

The continuity of $g^{(k)}$ away from zero follows from (ii).

Point (iv) is a straightforward consequence of the definition of S_{rad} . \square

Remark 5.4.4. *The gain of regularity in point (ii) cannot be extended to $[0, \infty)$. We can see this by taking $f(t) = t^{1/2}$.*

We now consider Bukhgeim solutions for the Schrödinger equation where the potential has been rotated. Using these solutions, we construct a new family of solutions $u_{\lambda,x,\theta}$ for the Schrödinger equation with the original potential (before taking the rotation). These new solutions depend on an additional parameter $\theta \in [0, 2\pi)$.

More precisely, let $x, z \in \mathbb{R}^2$ and let $R_{x,\theta}(z)$ denote the rotation of z around x by an angle θ , given by

$$R_{x,\theta}(z) = x + \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} (z - x).$$

Letting $q_{x,\theta}(z) := q \circ R_{x,\theta}(z)$, we consider Bukhgeim solutions

$$u_{\lambda,x} = e^{i\lambda\psi_x}(1 + w_{\lambda,x})$$

to the Schrödinger equation with rotated potential $\Delta u = q_{x,\theta} u$. We write

$$u_{\lambda,x,\theta}(z) := u_{\lambda,x} \circ R_{x,-\theta}(z),$$

and consider the boundary information at frequency λ defined by

$$BI_x(\lambda) := \frac{\lambda}{2\pi^2} \int_0^{2\pi} \left\langle (\Lambda_q - \Lambda_0)[u_{\lambda,x,\theta}|_{\partial\Omega}], e^{i\lambda\bar{\psi}_x \circ R_{x,-\theta}}|_{\partial\Omega} \right\rangle d\theta.$$

By the conformal invariance of the Laplacian,

$$\Delta u_{\lambda,x,\theta} = \Delta(u_{\lambda,x} \circ R_{x,-\theta}) = (\Delta u_{\lambda,x}) \circ R_{x,-\theta},$$

we see that the $u_{\lambda,x,\theta}$ solves the original Schrödinger equation $\Delta u = q u$. By the same rotational invariance, $e^{i\lambda\psi_x \circ R_{x,-\theta}}$ is a solution to Laplace's equation, and so we can still use Alessandrini's identity (2.2) to reinterpret the boundary information. Moreover $u_{\lambda,x,\theta}|_{\partial\Omega}$ can be recovered from $\Lambda_q - \Lambda_0$ by a suitably rotated version of Theorem 3.2.6.

Theorem 5.4.5. *Let $s > 0$, let $q \in H^s(\mathbb{R}^2)$ be a complex-valued potential and let Ω be a bounded Lipschitz domain in the plane. Then, for any x such that $q \in C^2(B(x, r))$ for some $r > 0$, we have*

$$\lim_{\lambda \rightarrow \infty} A_{\text{freq}}^3[BI_x](\lambda) = q(x).$$

Proof. Without loss of generality, we can suppose that we are recovering the potential at the origin, and so we omit the dependence on $x = 0$. The first

step is to split the reconstruction formula into a main term and a remainder term using Alessandrini's identity (2.2), to obtain

$$\begin{aligned}
BI_x(\lambda) &= \frac{\lambda}{2\pi^2} \int_0^{2\pi} \left\langle (\Lambda_q - \Lambda_0)[u_{\lambda,x,\theta}(z)], e^{i\lambda\bar{\psi} \circ R_{-\theta}}|_{\partial\Omega} \right\rangle d\theta \\
&= \frac{\lambda}{2\pi^2} \int_0^{2\pi} \int_{\mathbb{R}^2} q(z) e^{i\lambda\phi \circ R_{-\theta}(z)} (1 + w_\lambda \circ R_{-\theta}(z)) dz d\theta \\
&= \frac{\lambda}{2\pi^2} \int_0^{2\pi} \int_{\mathbb{R}^2} q \circ R_\theta(z) e^{i\lambda\phi(z)} (1 + w_\lambda(z)) dz d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} T^\lambda[q \circ R_\theta] + T_w^\lambda[q \circ R_\theta] d\theta.
\end{aligned}$$

Now, by Fubini's theorem we see that

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} T^\lambda[q \circ R_\theta] d\theta &= \frac{\lambda}{2\pi^2} \int_{\mathbb{R}^2} e^{i\lambda\phi(z)} \int_0^{2\pi} q \circ R_\theta(z) d\theta dz \\
&= T^\lambda[V_0]
\end{aligned}$$

where $V_0(z) = A_{\text{ang}}[q](|z|)$. Thus, we find

$$A_{\text{freq}}^3[BI_x](\lambda) = A_{\text{freq}}^3[T^{(\cdot)}[V_0]] + A_{\text{freq}}^3\left[\frac{1}{2\pi} \int_0^{2\pi} T_w^{(\cdot)}[q \circ R_\theta] d\theta\right].$$

Now by Lemma 5.4.1, we have that

$$A_{\text{freq}}^3[T^{(\cdot)}[V_0]](\lambda) = T^\lambda[V_3], \quad \text{where } V_j = S_{\text{rad}}[V_{j-1}],$$

and by Lemma 3.2.5 we know that the remainder satisfies

$$\lim_{\lambda \rightarrow \infty} T_w^\lambda[q \circ R_\theta] = 0.$$

Thus we find that

$$\lim_{\lambda \rightarrow \infty} A_{\text{freq}}^3[BI_x](\lambda) = \lim_{\lambda \rightarrow \infty} T^\lambda[V_3].$$

Noting that $V_3(0) = q(0)$, it remains to prove that $V_3 \in H^2(\mathbb{R}^2)$ so that we can conclude using Lemma 3.2.4.

To see that $V_3 \in H^2(\mathbb{R}^2)$, recall that $q \in H^s(\mathbb{R}^2)$ is compactly supported, so that $q \in L^1(\mathbb{R}^2)$, which gives us

$$\begin{aligned}
\int_0^1 |V_0(\rho, \theta)| \rho d\rho &= \int_0^1 \left| \frac{1}{2\pi} \int_0^{2\pi} q(\rho, \theta) d\theta \right| \rho d\rho \\
&\leq \int_0^1 \int_0^{2\pi} |q(\rho, \theta)| d\theta \rho d\rho < \infty
\end{aligned}$$

for any $\theta \in [0, 2\pi)$. By Lemma 5.4.2 we know that $V_0 \in C^2(B_r)$ so $V_0(\rho, \alpha)$ is bounded for $\rho < r/2$; therefore the one variable function $V_0(\rho)$ belongs to $L^1(\mathbb{R}^+)$. Let $\varphi \in C_0^\infty(B_r)$ be a radial function such that $\varphi(z) = 1$ for $|z| < r/2$. As $V_0 \in C^2(B_r)$ we can use part (iii) of Lemma 5.4.3 to obtain

$$S_{\text{rad}}[\varphi V_0] \in C_0^2(\mathbb{R}^2),$$

part (i) of Lemma 5.4.3 to gain regularity away from zero

$$S_{\text{rad}}[(1 - \varphi)V_0] \in C_0^0(\mathbb{R}^2),$$

and part (iv) of Lemma 5.4.3 to control the support

$$S_{\text{rad}}[(1 - \varphi)V_0(\cdot, \alpha)](\rho) = 0, \quad \text{for } \rho < r/2,$$

leading to

$$V_1 \in C_0^0(\mathbb{R}^2) \cap C^2(B_{r/2})$$

given that $V_1 = S_{\text{rad}}[\varphi V_0] + S_{\text{rad}}[(1 - \varphi)V_0]$. Using the same arguments (but using part (ii) of Lemma 5.4.3 to gain regularity instead of part (i) of Lemma 5.4.3) we can see that $V_2 \in C_0^1(\mathbb{R}^2) \cap C^2(B_{r/4})$ and finally $V_3 \in C_0^2(\mathbb{R}^2) \subset H^2(\mathbb{R}^2)$. \square

Chapter 6

Uniqueness for complex conductivities

6.1 Introduction

In this chapter we consider the Calderón's inverse problem in the plane. We show that complex-valued Lipschitz conductivities are uniquely determined by the DtN map. This is a joint work with Evgeny Lakshtanov and Boris Vainberg and was published in [47].

In particular, the conductivities γ that we consider are of the form $\gamma(z) = \sigma(z) + i\omega\epsilon(z)$, where σ is the electric conductivity and ϵ is the electric permittivity. In much of the literature, the frequency ω is considered negligibly small, so that γ is a real-valued function, however here we will avoid this approximation. The Dirichlet problem for the conductivity equation is to find u satisfying

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f \end{cases} \quad (6.1)$$

where Ω is a bounded Lipschitz domain in the plane and $f \in H^{1/2}(\partial\Omega)$. Supposing that $0 < c < \sigma < c^{-1}$, there is a unique solution and the DtN map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega) := (H^{1/2}(\partial\Omega))^*$$

can be defined via duality by

$$\langle \Lambda_\gamma[f], v|_{\partial\Omega} \rangle := \int_\Omega \gamma \nabla u \cdot \nabla v$$

for any $v \in H^1(\Omega)$. With sufficient regularity, this definition coincides with that of the introduction by integration by parts.

The main aim of this chapter will be to prove the following theorem.

Theorem 6.1.1. *Let $\gamma_1, \gamma_2 \in \text{Lip}(\overline{\Omega})$ be complex-valued conductivities. Then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2.$$

The result is based on a development of the Bukhgeim approach combined with some arguments of Brown and Uhlmann from [20].

Remark 6.1.2. *From now on we make the usual assumption that γ takes the value one near the boundary of the domain Ω . This assumption can be made because the conductivity can be recovered at the boundary (see [3] or [19]) and then extended to a larger domain using the method of Whitney such that the extended conductivity is equal to one close to the boundary of the larger domain. See [51, Section 6] for more details or [32, Section 2] for the case of Lipschitz conductivities.*

In the following section we give a sketch of the proof, with the details given in Section 6.3.

6.2 Main steps

6.2.1 Reduction to the Dirac equation

The following observation made in [20] plays an important role. Let u be a solution of (6.1). Then the vector

$$\varrho = \gamma^{1/2}(\partial u, \bar{\partial} u)^t$$

satisfies the Dirac equation

$$\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \varrho = q \varrho \quad (6.2)$$

where

$$q(z) = \begin{pmatrix} 0 & q_{12}(z) \\ q_{21}(z) & 0 \end{pmatrix}, \quad q_{12} = -\frac{1}{2}\partial \log \gamma, \quad q_{21} = -\frac{1}{2}\bar{\partial} \log \gamma. \quad (6.3)$$

If q is found and the conductivity γ is known at one point $z_0 \in \overline{\Omega}$, then γ in Ω can be immediately found from (6.3).

From now on, we use a different form of equation (6.2); instead of Beals-Coifmann notation $\varrho = (\varrho_1, \varrho_2)^t$, we rewrite the equation in Sung's notation: $v_1 = \varrho_1, v_2 = \overline{\varrho_2}$. Then the vector $v = (v_1, v_2)^t$ is a solution of the system

$$\bar{\partial} v = Q \bar{v} \quad (6.4)$$

where

$$Q(z) = \begin{pmatrix} 0 & q_{12}(z) \\ \overline{q_{21}}(z) & 0 \end{pmatrix}.$$

6.2.2 Solving the Dirac equation for large $|\lambda|$

Let v be a *matrix* solution of (6.4) that depends on parameter $\lambda \in \mathbb{C}$ and has the following behaviour at infinity

$$\lim_{|z| \rightarrow \infty} v_{\lambda,x}(z) e^{-\lambda(z-x)^2/2} = \mathbf{I}. \quad (6.5)$$

It is worth noting that, contrary to standard practice, we consider the function v (and other functions defined by v) for all complex values of λ , not just for $i\lambda$ with $\lambda > 0$. This allows us to generalize the Bukhgeim method to the case of potentials in $L^\infty(\mathbb{R}^2)$. From the technical point of view, this allows us to use the Hausdorff-Young inequality.

Problem (6.4)-(6.5) can be rewritten using a bounded function

$$\mu_{\lambda,x}(z) := v_{\lambda,x}(z) e^{-\lambda(z-x)^2/2}; \quad (6.6)$$

that is to say (6.4)-(6.5) is equivalent to

$$\bar{\partial} \mu_{\lambda,x}(z) = Q(z) \overline{\mu_{\lambda,x}}(z) e^{(\overline{\lambda(z-x)^2} - \lambda(z-x)^2)/2} \quad (6.7)$$

with $\lim_{|z| \rightarrow \infty} \mu = \mathbf{I}$. Let $\rho_{\lambda,x}$ be the real-valued function defined by

$$\rho_{\lambda,x}(z) = -i (\overline{\lambda(z-x)^2} - \lambda(z-x)^2)/2$$

and let \mathcal{L}_λ be the operator defined by

$$\mathcal{L}_\lambda[f](z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{e^{i\rho_{\lambda,x}(z')}}{z - z'} f(z') dz'.$$

Equation (6.7) can be reduced to the Lippmann-Schwinger equation

$$\mu_{\lambda,x}(z) = \mathbf{I} + \frac{1}{\pi} \int_{\mathbb{C}} Q(z') \frac{e^{i\rho_{\lambda,x}(z')}}{z - z'} \overline{\mu_{\lambda,x}}(z') dz', \quad (6.8)$$

where $\lim_{|z| \rightarrow \infty} \mu = \mathbf{I}$. The previous equation can be rewritten using \mathcal{L}_λ as

$$\mu = \mathbf{I} + \mathcal{L}_\lambda [Q + Q \overline{\mathcal{L}_\lambda} [\overline{Q} \mu]].$$

In particular, for each of the components we have

$$\begin{aligned} \mu_{11} &= 1 + \mathcal{L}_\lambda [q_{12} \overline{\mathcal{L}_\lambda} [q_{21} \mu_{11}]], \\ \mu_{12} &= \mathcal{L}_\lambda [q_{12}] + \mathcal{L}_\lambda [q_{12} \overline{\mathcal{L}_\lambda} [q_{21} \mu_{12}]], \\ \mu_{21} &= \mathcal{L}_\lambda [\overline{q_{21}}] + \mathcal{L}_\lambda [\overline{q_{21}} \overline{\mathcal{L}_\lambda} [\overline{q_{12}} \mu_{21}]], \\ \mu_{22} &= 1 + \mathcal{L}_\lambda [\overline{q_{21}} \overline{\mathcal{L}_\lambda} [\overline{q_{12}} \mu_{22}]]. \end{aligned}$$

Let $\mathcal{M}[f] = \mathcal{L}_\lambda [q_{12} \overline{\mathcal{L}_\lambda} [q_{21} f]]$. Then, for the component μ_{11} we have

$$(\mathbf{I} - \mathcal{M})[\mu_{11} - 1] = \mathcal{M}[1] \quad (6.9)$$

and for the component μ_{12} we have

$$(\mathbf{I} - \mathcal{M})[\mu_{12} - \mathcal{L}_\lambda[q_{12}]] = \mathcal{M} \circ \mathcal{L}_\lambda[q_{12}].$$

Switching the roles of q_{12} and q_{21} we can obtain similar expressions for the other elements of the matrix μ . By inverting $\mathbf{I} - \mathcal{M}$, we can obtain each of the components of μ . We work in the following space

$$L_{z,x}^\infty L_{|\lambda|>R}^p$$

consisting of bounded functions of $z, x \in \mathbb{C}$ with values in the Banach space $L_{|\lambda|>R}^p$, defined by

$$L_{|\lambda|>R}^p = L_\lambda^p(\lambda : |\lambda| > R).$$

The following lemmas show that $\mathcal{M} : L_{z,x}^\infty L_{|\lambda|>R}^p \rightarrow L_{z,x}^\infty L_{|\lambda|>R}^p$ is a contractive operator if R is large enough, and that $\mathcal{M}[1]$ and $\mathcal{M}[\mathcal{L}_\lambda q_{12}]$ belong to $L_{z,x}^\infty L_{|\lambda|>R}^p$. Thus, one can find the solution μ of (6.8) using Neumann series to invert $(\mathbf{I} - \mathcal{M})$.

Lemma 6.2.1. *Let $p > 2$. Then*

$$\lim_{R \rightarrow \infty} \|\mathcal{M}\|_{L_{z,x}^\infty L_{|\lambda|>R}^p \rightarrow L_{z,x}^\infty L_{|\lambda|>R}^p} = 0.$$

Lemma 6.2.2. *Let $p > 2$. Then there exists $R > 0$ such that*

$$\mathcal{M}[1] \in L_{z,x}^\infty L_{|\lambda|>R}^p.$$

Lemma 6.2.3. *Let $p > 2$. Then there exists $R > 0$ such that*

$$\mathcal{M} \circ \mathcal{L}_\lambda[q_{12}] \in L_{z,x}^\infty L_{|\lambda|>R}^p.$$

Note that (6.9) together with Lemmas 6.2.1, 6.2.2 and 6.2.3 allows one to solve the direct but not the inverse problem, since the operator \mathcal{M} depends on Q . That is, for the purpose of reconstruction a different characterization of μ , in terms of the DtN map, would be required.

6.2.3 Determination of the potential

For g a 2×2 matrix-valued function let \mathcal{T}^λ be the operator defined by

$$\mathcal{T}^\lambda[g] = \int_\Omega e^{i\rho_{\lambda,x}} Q g.$$

We define the (*generalized*) *scattering data* as the matrix h given by

$$h(\lambda, x) = \mathcal{T}^\lambda[\overline{\mu_{\lambda,x}}]. \quad (6.10)$$

By (6.7) we have $h = \int_{\Omega} \bar{\partial} \mu_{\lambda,x}$ so we can use Green's formula

$$\int_{\Omega} \bar{\partial} f = \frac{1}{2i} \int_{\partial\Omega} \nu f$$

to rewrite h as

$$h(\lambda, x) = \frac{1}{2i} \int_{\partial\Omega} \nu \mu_{\lambda,x} \quad (6.11)$$

where (ν_1, ν_2) is the exterior normal vector to $\partial\Omega$ and $\nu = \nu_1 + i\nu_2$. Thus, one does not need to know the potential Q in order to find h . The function h can be evaluated if the Dirichlet data $v|_{\partial\Omega}$ is known for the equation (6.4), since $\mu|_{\partial\Omega}$ in (6.11) can be expressed via $v|_{\partial\Omega}$ using (6.6).

The spectral parameter $i\lambda$ with real λ was used in the standard Bukhgeim approach to recover the potential from scattering data (6.10), and the potential was recovered by the limit of the scattering data as $\lambda \rightarrow \infty$. Instead, in the present chapter, we have $\lambda \in \mathbb{C}$, and the potential is determined by integrating the scattering data over a large annulus in the complex λ -plane, which we denote by

$$A_R = \{\lambda \in \mathbb{C} : R < |\lambda| < 2R\}.$$

For a matrix M , let M^d denote the matrix with the entries in the main diagonal equal to those of M and with the remaining entries equal to zero, and let M^o denote the matrix with entries off the main diagonal equal to those of M and with the entries in the main diagonal equal to zero.

The next lemma is useful in the proof of the theorem below. The theorem indicates how Q is uniquely determined by the scattering data h .

Lemma 6.2.4. *Let $p > 1$. Then there exists $R > 0$ such that*

$$\mathcal{T}^\lambda[\bar{\mu}^d - \mathbb{I}] \in L_x^\infty L_{|\lambda|>R}^p.$$

Theorem 6.2.5. *Let Q be a complex-valued bounded potential. Then for any $g \in C_0^\infty(\Omega)$ we have*

$$\frac{1}{2\pi^2 \ln 2} \lim_{R \rightarrow \infty} \int_{A_R} |\lambda|^{-1} \int_{\Omega} g(x) h^o(\lambda, x) dx d\lambda = \int_{\Omega} g(z) Q(z) dz.$$

6.3 Proofs

6.3.1 Preliminary results

Before stepping into the proofs of the previously stated results, we introduce some auxiliary results.

Lemma 6.3.1. *Let $\lambda, x \in \mathbb{C}$ and let $\varphi \in L^1(\mathbb{C})$. Then*

$$\lim_{|\lambda| \rightarrow \infty} \int_{\mathbb{C}} \varphi(z) e^{i\rho_{\lambda,x}(z)} dz = 0$$

uniformly in x .

Proof. Let us approximate φ by a sequence $\varphi_n \in C_0^\infty$ in the L^1 norm. For any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\left| \int_{\mathbb{C}} \varphi(z) e^{i\rho_{\lambda,x}(z)} dz - \int_{\mathbb{C}} \varphi_N(z) e^{i\rho_{\lambda,x}(z)} dz \right| \leq \int_{\mathbb{C}} |\varphi(z) - \varphi_N(z)| dz \leq \varepsilon.$$

Now, using the stationary phase method [60, Chapter VIII, Proposition 5] there exists a constant C , independent of λ, φ_N and x , such that

$$\begin{aligned} \left| \int_{\mathbb{C}} \varphi_N(z) e^{i\rho_{\lambda,x}(z)} dz \right| &= \left| \int_{\mathbb{C}} \varphi_N(z+x) e^{i\rho_{\lambda,0}(z)} dz \right| \\ &\leq C \lambda^{-1/2} (\|\varphi_N\|_{L^\infty} + \|\nabla \varphi_N\|_{L^1}) \end{aligned}$$

and the proof is concluded. \square

Lemma 6.3.2. *Let $1 \leq p < 2$ and let $R > 0$. Then, for $a \in \mathbb{C} \setminus \{0\}$, there exists constants $C = C(p, R)$ and $\delta = \delta(p) > 0$ such that*

$$\|u^{-1}(\sqrt{u} - a)^{-1}\|_{L^p(B_R)} \leq C(1 + |a|^{-1+\delta})$$

where B_R is the ball of radius R .

Proof. The statement is obvious if $|a| \geq 1$. If $|a| < 1$, we make the substitution $u = |a|^2 v$ and $\dot{a} = a|a|^{-1}$ to obtain

$$\|u^{-1}(\sqrt{u} - a)^{-1}\|_{L^p(B_R)} = |a|^{\frac{4}{p}-3} \|v^{-1}(\sqrt{v} - \dot{a})^{-1}\|_{L^p(B_{R'})}$$

with $R' = R|a|^{-2}$. We now split the domain to see

$$\|v^{-1}(\sqrt{v} - \dot{a})^{-1}\|_{L^p(B_2)} \leq C$$

and

$$\|v^{-1}(\sqrt{v} - \dot{a})^{-1}\|_{L^p(B_{R'} \setminus B_2)} \leq 4 \left\| |v|^{-3/2} \right\|_{L^p(B_{R'} \setminus B_2)}.$$

Using polar coordinates we can compute the right-hand side

$$\left\| |v|^{-3/2} \right\|_{L^p(B_{R'} \setminus B_2)} = \left(2\pi \int_2^{R'} t^{1-3p/2} dt \right)^{1/p}$$

which is bounded by a constant if $p > 4/3$, by $C(1 + |\ln |a||^{3/4})$ if $p = 4/3$, and by $C|a|^{3-\frac{4}{p}}$ if $p < 4/3$. \square

Lemma 6.3.3. *Let $z, x \in \mathbb{C}$, $p > 2$ and $\varphi \in L_0^\infty(\mathbb{C})$. Then there exists a constant C , which depends only on the support of φ and on $\delta = \delta(p) > 0$, such that*

$$\left\| \int_{\mathbb{C}} \frac{e^{i\rho_{\lambda,x}(z_1)}}{z - z_1} \varphi(z_1) dz_1 \right\|_{L_\lambda^p(\mathbb{C})} \leq C \frac{\|\varphi\|_{L^\infty}}{|z - x|^{1-\delta}}.$$

Proof. Let

$$f(z, \lambda, x) = \int_{\mathbb{C}} \frac{e^{i\rho_{\lambda,x}(z_1)}}{z - z_1} \varphi(z_1) dz_1.$$

Making a change of variables $u = (z_1 - x)^2$ in f and taking into account that $du = 4|z_1 - x|^2 dz_1$ we have

$$f = \frac{1}{4} \sum_{\pm} \int_{\mathbb{C}} \frac{e^{i(\bar{\lambda}u - \lambda u)/2}}{|u|(\mp\sqrt{u} + (z - x))} \varphi(x \pm \sqrt{u}) du.$$

Using the Hausdorff-Young inequality with $p' = p/(p-1)$ and Lemma 6.3.2

$$\|f\|_{L_\lambda^p} \leq \frac{1}{2} \sum_{\pm} \left\| \frac{\varphi(x \pm \sqrt{u})}{|u|(\mp\sqrt{u} + (z - x))} \right\|_{L_u^{p'}} \leq C \frac{\|\varphi\|_{L^\infty}}{|z - x|^{1-\delta}}$$

and the proof is concluded. \square

Proof of Lemma 6.2.1

Let $g_{\lambda,x}(z) \in L_{z,x}^\infty L_{|\lambda|>R}^p$ and let

$$f(z, z_2, \lambda, x) = \int_{\Omega} \frac{e^{i\rho_{\lambda,x}(z_1)}}{z - z_1} q_{12}(z_1) \frac{e^{-i\rho_{\lambda,x}(z_2)}}{\bar{z}_1 - \bar{z}_2} q_{21}(z_2) dz_1,$$

so that, using Fubini's theorem, we have

$$\mathcal{M}[g_{\lambda,x}](z) = \int_{\Omega} f(z, z_2, \lambda, x) g_{\lambda,x}(z_2) dz_2.$$

Then, from Minkowski's integral inequality and Hölder's inequality, we have

$$\begin{aligned} \|\mathcal{M}[g_{\lambda,x}](z)\|_{L_{|\lambda|>R}^p} &\leq \int_{\Omega} \|f(z, z_2, \lambda, x) g_{\lambda,x}(z_2)\|_{L_{|\lambda|>R}^p} dz_2 \\ &\leq \sup_{z_2} \|g_{\lambda,x}(z_2)\|_{L_{|\lambda|>R}^p} \int_{\Omega} \sup_{\lambda: |\lambda|>R} |f(z, z_2, \lambda, x)| dz_2. \end{aligned}$$

Thus it remains to show that uniformly in $z \in \mathbb{C}$ and $x \in \Omega$ we have

$$\lim_{|\lambda| \rightarrow \infty} \int_{\Omega} |f(z, z_2, \lambda, x)| dz_2 = 0.$$

Using Fubini's theorem we can see

$$\int_{B_1(0)} \int_{B_1(0)} \frac{1}{|z_1|} \frac{1}{|z_1 - z_2|} dz_1 dz_2 = \int_{B_1(0)} \frac{1}{|z_1|} \int_{B_1(0)} \frac{1}{|z_1 - z_2|} dz_2 dz_1 < \infty.$$

Then for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that

$$\int_{B_\delta(z)} |f(z, z_2, \lambda, x)| dz_2 \leq \|Q\|_{L^\infty}^2 \int_{B_\delta(z)} \int_{\Omega} \frac{1}{|z - z_1|} \frac{1}{|z_1 - z_2|} dz_1 dz_2 < \varepsilon.$$

For $z_2 \in \Omega \setminus B_\delta(z)$ we can use Lemma 6.3.1 to see that

$$\lim_{|\lambda| \rightarrow \infty} f(z, z_2, \lambda, x) = 0 \quad (6.12)$$

uniformly in x . Note that f is the Cauchy transform of g on z , given by

$$g(z, z_2, \lambda, x) = e^{i\rho_{\lambda, x}(z)} q_{12}(z) \frac{e^{-i\rho_{\lambda, x}(z_2)}}{\bar{z} - \bar{z}_2} q_{21}(z_2)$$

and we have $g \in L_z^{p'}$ with $2 < p' < \infty$ given that $z_2 \in \Omega \setminus B_\delta(z)$. Thus, by [8, Theorem 4.3.13] we have

$$\|f\|_{C_z^\alpha} \leq \frac{12p'^2}{p' - 2} \|g\|_{L_z^{p'}} \lesssim \|Q\|_{L^\infty}^2 \delta^{-1}$$

with $\alpha = 1 - 2/p'$, and where the implicit constant depends only on Ω and p' . Therefore f is equicontinuous with respect to z , and convergence in (6.12) is also uniform with respect to this variable. \square

Proof of Lemma 6.2.2

Applying successively Minkowski's integral inequality, Hölder's inequality and Lemma 6.3.3, we have

$$\begin{aligned} \|\mathcal{M}[1]\|_{L_{|\lambda|>R}^p} &\leq \int_{\Omega} \left\| \frac{e^{i\rho_{\lambda}(z_1)}}{z - z_1} q_{12}(z_1) \int_{\Omega} \frac{e^{-i\rho_{\lambda}(z_2)}}{\bar{z}_1 - \bar{z}_2} q_{21}(z_2) dz_2 \right\|_{L_{|\lambda|>R}^p} dz_1 \\ &\leq \int_{\Omega} \left| \frac{q_{12}(z_1)}{z - z_1} \right| \left\| \int_{\Omega} \frac{e^{-i\rho_{\lambda}(z_2)}}{\bar{z}_1 - \bar{z}_2} q_{21}(z_2) dz_2 \right\|_{L_{|\lambda|>R}^p} dz_1 \\ &\leq C \int_{\Omega} \frac{1}{|z - z_1| |z_1 - x|^{1-\delta}} dz_1 < \infty, \end{aligned}$$

since $\delta > 0$, and where C is a constant depending on $\|Q\|_{L^\infty}$ and Ω . \square

Proof of Lemma 6.2.3

Let f be given by

$$f(z, \lambda, x) = \pi^{-2} \int_{\Omega} \frac{e^{-i\rho_{\lambda}(z_1)}}{\bar{z} - \bar{z}_1} q_{21}(z_1) \int_{\Omega} \frac{e^{i\rho_{\lambda}(z_2)}}{z_1 - z_2} q_{12}(z_2) dz_2 dz_1.$$

Then we have

$$\mathcal{M} \circ \mathcal{L}_{\lambda}[q_{12}](z) = \pi^{-1} \int_{\Omega} \frac{e^{i\rho_{\lambda}(z_1)}}{z - z_1} q_{12}(z_1) f(z_1, \lambda, x) dz_1.$$

By Minkowski's integral inequality and Hölder's inequality we have

$$\begin{aligned} \|\mathcal{M} \circ \mathcal{L}_{\lambda}[q_{12}]\|_{L^p_{|\lambda|>R}} &\leq \int_{\Omega} \left\| \frac{e^{i\rho_{\lambda}(z_1)}}{z - z_1} q_{12}(z_1) f(z_1, \lambda, x) \right\|_{L^p_{|\lambda|>R}} dz_1 \\ &\leq \|f\|_{L^p_{|\lambda|>R}} \int_{\Omega} \left| \frac{q_{12}(z_1)}{z - z_1} \right| dz_1. \end{aligned}$$

Applying successively Minkowski's integral inequality, Hölder's inequality and Lemma 6.3.3, we have

$$\begin{aligned} \|f\|_{L^p_{|\lambda|>R}} &\leq \int_{\Omega} \left\| \frac{e^{-i\rho_{\lambda}(z_1)}}{\bar{z} - \bar{z}_1} q_{21}(z_1) \int_{\Omega} \frac{e^{i\rho_{\lambda}(z_2)}}{z_1 - z_2} q_{12}(z_2) dz_2 \right\|_{L^p_{|\lambda|>R}} dz_1 \\ &\leq \int_{\Omega} \left| \frac{q_{21}(z_1)}{\bar{z} - \bar{z}_1} \right| \left\| \int_{\Omega} \frac{e^{i\rho_{\lambda}(z_2)}}{z_1 - z_2} q_{12}(z_2) dz_2 \right\|_{L^p_{|\lambda|>R}} dz_1 \\ &\leq C \int_{\Omega} \frac{1}{|z - z_1| |z_1 - x|^{1-\delta}} dz_1 < \infty, \end{aligned}$$

since $\delta > 0$, and where C is a constant depending on $\|Q\|_{L^\infty}$ and Ω . \square

Proof of Lemma 6.2.4

Note that $\mathcal{T}^\lambda[\bar{\mu}^d - \mathbf{I}]$ is a matrix with zeros on the diagonal. The entry $(\mathcal{T}^\lambda[\bar{\mu}^d - \mathbf{I}])_{21}$ depends only on μ_{11} ; we perform the computations for this entry. The other entry follows from the same arguments. We start with the relation

$$\mu_{11} - 1 = \mathcal{M}[\mu_{11} - 1] + \mathcal{M}[1]$$

and this leads us to

$$(\mathcal{T}^\lambda[\bar{\mu}^d - \mathbf{I}])_{21} = \int_{\Omega} e^{i\rho_{\lambda,x}} \bar{q}_{21} \mathcal{M}[\mu_{11} - 1] + \int_{\Omega} e^{i\rho_{\lambda,x}} \bar{q}_{21} \mathcal{M}[1].$$

Let $f = \overline{\mu_{11}} - 1$. Applying successively Fubini's theorem, Minkowski's integral inequality, Holder's inequality, and Lemma 6.3.3, we see that

$$\begin{aligned}
& \left\| \int_{\Omega} e^{i\rho_{\lambda,x}(z)} \overline{q_{21}}(z) \mathcal{M}[f](z) dz \right\|_{L^p_{|\lambda|>R}} \\
& \leq C \int_{\Omega} \left\| \int_{\Omega} \frac{e^{i\rho_{\lambda,x}(z)}}{z - z_1} \overline{q_{21}}(z) dz \right\|_{L^{2p}_{|\lambda|>R}} \left\| \int_{\Omega} \frac{e^{-i\rho_{\lambda,x}(z_2)}}{\overline{z_1} - \overline{z_2}} q_{21}(z_2) f(z_2) dz_2 \right\|_{L^{2p}_{|\lambda|>R}} dz_1 \\
& \leq C^2 \int_{\Omega} \left\| \int_{\Omega} \frac{e^{i\rho_{\lambda,x}(z)}}{z - z_1} \overline{q_{21}}(z) dz \right\|_{L^{2p}_{|\lambda|>R}} \int_{\Omega} \left| \frac{q_{21}(z_2)}{\overline{z_1} - \overline{z_2}} \right| \|f(z_2)\|_{L^{2p}_{|\lambda|>R}} dz_2 dz_1 \\
& \leq C^3 \|f\|_{L^{2p}_{|\lambda|>R}} \int_{\Omega} \frac{1}{|z_1 - x|^{1-\delta}} dz_1 < \infty,
\end{aligned}$$

since $\delta > 0$ and $f \in L^{\infty}_{z,x} L^{2p}_{|\lambda|>R}$ (due to (6.9) and lemmas 6.2.1 and 6.2.2), and where C is a constant depending on $\|Q\|_{L^{\infty}}$ and Ω .

For the other term we follow similarly. Applying successively Fubini's theorem, Minkowski's integral inequality, Holder's inequality, and Lemma 6.3.3, we see that

$$\begin{aligned}
& \left\| \int_{\Omega} e^{i\rho_{\lambda,x}(z)} \overline{q_{21}}(z) \mathcal{M}[1](z) dz \right\|_{L^p_{|\lambda|>R}} \\
& \leq \int_{\Omega} \left\| \int_{\Omega} \frac{e^{i\rho_{\lambda,x}(z)}}{z - z_1} \overline{q_{21}}(z) dz \int_{\Omega} \frac{e^{-i\rho_{\lambda,x}(z_2)}}{\overline{z_1} - \overline{z_2}} q_{21}(z_2) dz_2 \right\|_{L^p_{|\lambda|>R}} |e^{i\rho_{\lambda,x}(z_1)} q_{12}(z_1)| dz_1 \\
& \leq C \int_{\Omega} \left\| \int_{\Omega} \frac{e^{i\rho_{\lambda,x}(z)}}{z - z_1} \overline{q_{21}}(z) dz \right\|_{L^{2p}_{|\lambda|>R}} \left\| \int_{\Omega} \frac{e^{-i\rho_{\lambda,x}(z_2)}}{\overline{z_1} - \overline{z_2}} q_{21}(z_2) dz_2 \right\|_{L^{2p}_{|\lambda|>R}} dz_1 \\
& \leq C^2 \int_{\Omega} \frac{1}{|z_1 - x|^{1-\delta}} \frac{1}{|z_1 - x|^{1-\delta}} dz_1 < \infty,
\end{aligned}$$

since $\delta > 0$, and where C is a constant depending on $\|Q\|_{L^{\infty}}$ and Ω . \square

Proof of Theorem 6.2.5

First note that

$$h^o = \mathcal{T}^{\lambda} [\overline{\mu}^d] = \mathcal{T}^{\lambda} [\mathbf{I}] + \mathcal{T}^{\lambda} [\overline{\mu}^d - \mathbf{I}]$$

so we can write

$$\begin{aligned}
\int_{A_R} |\lambda|^{-1} \int_{\Omega} g(x) h^o(\lambda, x) dx d\lambda &= \int_{A_R} |\lambda|^{-1} \int_{\Omega} g(x) \mathcal{T}^{\lambda} [\mathbf{I}] dx d\lambda \\
&\quad + \int_{A_R} |\lambda|^{-1} \int_{\Omega} g(x) \mathcal{T}^{\lambda} [\overline{\mu}^d - \mathbf{I}] dx d\lambda.
\end{aligned}$$

For the main term, first note that using the stationary phase method [60, Chapter VIII, Proposition 6] for $|\lambda|$ large enough we have

$$\int_{\Omega} g(x) e^{i\rho_{\lambda,x}(z)} dx = \pi |\lambda|^{-1} g(z) + O(|\lambda|^{-2})$$

which leads to

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{A_R} |\lambda|^{-1} \int_{\Omega} g(x) e^{i\rho_{\lambda,x}(z)} dx d\lambda &= \lim_{R \rightarrow \infty} \int_{A_R} \pi |\lambda|^{-2} g(z) + O(|\lambda|^{-3}) d\lambda \\ &= 2\pi^2 \ln 2 g(z). \end{aligned}$$

Then, by Fubini's theorem and the dominated convergence theorem we have

$$\begin{aligned} &\lim_{R \rightarrow \infty} \int_{A_R} |\lambda|^{-1} \int_{\Omega} g(x) \mathcal{T}^{\lambda}[\mathbf{I}] dx d\lambda \tag{6.13} \\ &= \lim_{R \rightarrow \infty} \int_{\Omega} \int_{A_R} |\lambda|^{-1} \int_{\Omega} g(x) e^{i\rho_{\lambda,x}(z)} Q(z) dx d\lambda dz \\ &= 2\pi^2 \ln 2 \int_{\Omega} g(z) Q(z) dz. \end{aligned}$$

Now, for the remainder term we use Holder's inequality

$$\int_{A_R} \left| |\lambda|^{-1} \mathcal{T}^{\lambda}[\bar{\mu}^d - \mathbf{I}] \right| d\lambda \leq \left(\int_{A_R} |\lambda|^{-q} d\lambda \right)^{1/q} \|\mathcal{T}^{\lambda}[\bar{\mu}^d - \mathbf{I}]\|_{L^p_{|\lambda|>R}}.$$

Taking $1 < p < 2$ we can use Lemma 6.2.4 to see that

$$\mathcal{T}^{\lambda}[\bar{\mu}^d - \mathbf{I}] \in L_x^{\infty} L_{|\lambda|>R}^p$$

and given that $q > 2$ we have

$$\lim_{R \rightarrow \infty} \left(\int_{A_R} |\lambda|^{-q} d\lambda \right)^{1/q} = 0.$$

Then, by Fubini's theorem and the dominated convergence theorem

$$\begin{aligned} &\lim_{R \rightarrow \infty} \int_{A_R} |\lambda|^{-1} \int_{\mathbb{C}} g(x) \mathcal{T}^{\lambda}[\bar{\mu}^d - \mathbf{I}] dx d\lambda \tag{6.14} \\ &= \lim_{R \rightarrow \infty} \int_{\mathbb{C}} g(x) \int_{A_R} |\lambda|^{-1} \mathcal{T}^{\lambda}[\bar{\mu}^d - \mathbf{I}] d\lambda dx = 0. \end{aligned}$$

Combining (6.14) and (6.13) completes the proof. \square

6.3.2 Proof of Theorem 6.1.1

Let γ_1, γ_2 be two Lipschitz conductivities in Ω such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, and let $Q^{(j)}, v^{(j)}, \mu^{(j)}, h^{(j)}, j = 1, 2$, be the potential and the solution in (6.4), the function in (6.6) and the scattering data in (6.10) for the conductivity γ_j . We follow the strategy used in [20, Theorem 4.1] and [31, Theorem 5.1]. The main idea of the proof is that, given that the two DtN maps coincide, the solutions $v^{(1)}$ and $v^{(2)}$ are equal in $\mathbb{C} \setminus \Omega$, and therefore the scattering data coincide. Then, using Theorem 6.2.5 we know that the potential $Q^{(j)}$ is uniquely determined by the scattering data $h^{(j)}$ for $|\lambda|$ large enough, and therefore so is γ_j .

Recall from Remark 6.1.2 that we can assume that the conductivities are equal to one close to the boundary of the domain. We extend the conductivities to be equal to one outside the domain.

Due to equation (6.11), we have

$$h^{(j)}(\lambda, w) = \frac{1}{2i} \int_{\partial\Omega} \mu^{(j)}(z, \lambda, w) dz.$$

Thus it is enough to prove that $\mu^{(1)}|_{\partial\Omega} = \mu^{(2)}|_{\partial\Omega}$, or equivalently

$$v^{(1)}|_{\partial\Omega} = v^{(2)}|_{\partial\Omega}, \quad (6.15)$$

for $|\lambda|$ large enough.

Let $\varphi^{(1)}$ denote the first column of $v^{(1)}$. Also let

$$\eta = \gamma_1^{-1/2} v_{11}^{(1)}, \quad \omega = \gamma_1^{-1/2} \overline{v_{21}^{(1)}}.$$

Since $\varphi^{(1)}$ satisfies

$$\bar{\partial}\varphi^{(1)} = Q^{(1)}\overline{\varphi^{(1)}}$$

then it follows that

$$\bar{\partial}\eta = -(\eta\bar{\partial}\gamma_1 + \omega\partial\gamma_1)(2\gamma_1)^{-1} = \partial\omega$$

and there exists u_1 such that

$$\partial u_1 = \eta, \quad \bar{\partial} u_1 = \omega \quad \text{in } \mathbb{C},$$

which is a solution to

$$\nabla \cdot (\gamma_1 \nabla u_1) = 0 \quad \text{in } \mathbb{C}.$$

Now we define u_2 by

$$u_2 = \begin{cases} u_1 & \text{in } \mathbb{C} \setminus \Omega \\ \tilde{u} & \text{in } \Omega, \end{cases}$$

where \tilde{u} is the solution to the Dirichlet problem

$$\begin{cases} \nabla \cdot (\gamma_2 \nabla \tilde{u}) = 0 & \text{in } \Omega \\ \tilde{u}|_{\partial\Omega} = u_1|_{\partial\Omega}. \end{cases}$$

Let $g \in C_0^\infty(\mathbb{C})$. Then

$$\begin{aligned} \int_{\mathbb{C}} \gamma_2 \nabla u_2 \cdot \nabla g \, dz &= \int_{\mathbb{C} \setminus \Omega} \gamma_1 \nabla u_1 \cdot \nabla g \, dz + \int_{\Omega} \gamma_2 \nabla \tilde{u} \cdot \nabla g \, dz \\ &= - \int_{\partial\Omega} \Lambda_{\gamma_1}[u_1|_{\partial\Omega}] g \, dz + \int_{\partial\Omega} \Lambda_{\gamma_2}[\tilde{u}|_{\partial\Omega}] g \, dz = 0. \end{aligned}$$

Hence

$$\nabla \cdot (\gamma_2 \nabla u_2) = 0 \quad \text{in } \mathbb{C}$$

and $\varphi^{(2)} = (\partial u_2, \partial \overline{u_2})^t$ satisfies

$$\overline{\partial} \varphi^{(2)} = Q^{(2)} \overline{\varphi^{(2)}}.$$

Lemmas 6.2.1, 6.2.2 and 6.2.3 imply the unique solvability of the Lippmann-Schwinger equation when $|\lambda| > R$ and R is large enough. Thus, $\varphi^{(2)}$ is equal to the first column of $v^{(2)}$ when $|\lambda| > R$. On the other hand, $\varphi^{(2)}$ in $\mathbb{C} \setminus \Omega$ coincides with $\varphi^{(1)}$, the first column of $v^{(1)}$. Thus the first columns of $v^{(1)}$ and $v^{(2)}$ are equal on $\mathbb{C} \setminus \Omega$ when $|\lambda| > R$. Repeating the same steps with the second columns of $v^{(1)}, v^{(2)}$, we obtain that $v^{(1)}|_{\partial\Omega} = v^{(2)}|_{\partial\Omega}$ when $|\lambda| > R$, and therefore (6.15) holds.

The uniqueness of h and Theorem 6.2.5 imply that the potential Q in the Dirac equation (6.4) is defined uniquely. Now the conductivity γ is uniquely determined by (6.3) up to an additive constant. Finally, this constant is uniquely determined since $\gamma|_{\partial\Omega}$ can be computed from Λ_γ . \square

Chapter 7

Numerical experiments with the Bukhgeim method

7.1 Introduction

In this chapter we perform some numerical experiments on the reconstruction method based on quadratic phase solutions. Numerical implementation of Theorem 3.2.6 is beyond the scope of this thesis; we approximate the Bukhgeim solutions on the boundary, or we start directly with the oscillatory integral inside the domain omitting the remainder term $w_{\lambda,x}$. In the latter case, our experiments consider the standard formula of Bukhgeim and those involving the averaging procedures introduced in Chapter 5.

In Section 7.2 we give a naive estimation on how measurement errors on the boundary are amplified and numerical instabilities worsen as λ grows. In Section 7.3 we reconstruct from the boundary data, using a first order approximation of the Bukhgeim solutions $u_{\lambda,x} = e^{i\lambda\psi_x}$; rather than proposing an alternative reconstruction method we want to study possible numerical instabilities and the tolerance to measurement errors.

Let q_λ denote the recovery of q by boundary measurements using the standard Bukhgeim reconstruction formula at frequency λ

$$q_\lambda(x) := \frac{\lambda}{\pi} \int_{\partial\Omega} (\Lambda_q - \Lambda_0)[u_{\lambda,x}] e^{i\lambda\bar{\psi}_x}.$$

By Alessandrini's identity and splitting into a *main term* and a *remainder term* we have

$$q_\lambda(x) = T^\lambda[q] + T_w^\lambda[q]$$

where

$$T^\lambda[q] = \int e^{i\lambda\phi_x(z)} q(z) dz, \quad \text{with} \quad \phi_x(z) := (z_1 - x_1)^2 - (z_2 - x_2)^2,$$

and the remainder term is defined similarly as in (3.8). In Section 7.4 we see how $T^\lambda[q]$ converges to the potential. Finally in Section 7.5 we study how the averaging strategies introduced in Chapter 5 improve the convergence of the main term of the approximation. In Section 7.6, we discuss our conclusions.

7.2 A peek into the error

We start by assuming that Ω is the unit ball. The reconstruction at frequency λ satisfies

$$\begin{aligned} q_\lambda &= \frac{\lambda}{\pi} \int_{\partial\Omega} (\Lambda_q - \Lambda_0)[u_\lambda] e^{i\lambda\bar{\psi}} \\ &= \frac{\lambda}{\pi} \int_{\partial\Omega} \Lambda_q[u_\lambda] e^{i\lambda\bar{\psi}} - \Lambda_0[e^{i\lambda\bar{\psi}}] u_\lambda \\ &= \frac{\lambda}{\pi} \int_{\partial\Omega} \Lambda_q[u_\lambda] e^{i\lambda\bar{\psi}} - u_\lambda \partial_\nu e^{i\lambda\bar{\psi}} \end{aligned}$$

where ∂_ν denotes the outward normal derivative on the boundary. In order to compute the integral in the boundary, we approximate it by a finite sum of evaluations of the integrand on a regular mesh on the boundary

$$q_{\lambda,N} = \frac{2\lambda}{N} \sum_{j=1}^N \Lambda_q[u_\lambda](z_j) e^{i\lambda\bar{\psi}(z_j)} - u_\lambda(z_j) \partial_\nu e^{i\lambda\bar{\psi}(z_j)}$$

where the points in the regular mesh are given by

$$z_j = (\cos(2\pi j/N), \sin(2\pi j/N)).$$

The DtN map is not measured with arbitrary precision, instead we deal with Λ_q^* , a noisy measurement of Λ_q . It is interesting to see how this noise propagates in the computation of q_λ . Given that the Bukhgeim solutions are obtained from the DtN map, there will be an error in their computation. Let u_λ^* denote the noise computation of u_λ which we model as

$$u_\lambda^* = u_\lambda + e_1$$

where e_1 is error term, which we model as a Gaussian white noise with standard deviation ϵ . We model the error in the outward normal derivative of u_λ in a similar way

$$\Lambda_q^*[u_\lambda^*] = \Lambda_q[u_\lambda] + e_2$$

where e_2 is a Gaussian white noise, and for simplicity we assume independent to e_1 and with the same standard deviation ϵ .

Let $q_{\lambda,N}^*$ denote the approximation obtained by discretizing the integral and considering the error from the measurement of the DtN map. As the

error in each point is a normal variable independent of the error in the other points, the variance of the sum is equal to the sum of the variances. Thus, the standard deviation of the error of the approximation $q_{\lambda,N}^*$ satisfies

$$\begin{aligned}\sigma(q_\lambda - q_{\lambda,N}^*) &= \frac{2\lambda}{N} \left(\sum_{j=1}^N \left(\epsilon |e^{i\lambda\bar{\psi}(z_j)}| \right)^2 + \left(\epsilon |\partial_\nu e^{i\lambda\bar{\psi}(z_j)}| \right)^2 \right)^{1/2} \\ &= \frac{2\lambda}{N} \left(\sum_{j=1}^N \epsilon^2 e^{2\lambda|\operatorname{Im} \psi(z_j)|} + \epsilon^2 \lambda^2 e^{2\lambda|\operatorname{Im} \psi(z_j)|} \right)^{1/2}\end{aligned}$$

where Im denotes the imaginary part.

The value of the previous expression could be made arbitrarily small by taking a thinner mesh to estimate the integral. However in practice this cannot be achieved, as the number of data source points cannot be increased arbitrarily. Also, and perhaps more relevant, is the fact that the decrease in the error as N increases is due to the independence assumption between the errors at each point, but this hypothesis does not seem to be realistic as, for example, one would expect that u_λ^* is continuous.

We obtain the following straightforward bound for the standard deviation of the error by taking only the largest term in the sum

$$\sigma(q_\lambda - q_{\lambda,N}^*) \leq \frac{2}{N} \epsilon \lambda^2 e^{\lambda\alpha_x}$$

where $\alpha_x = \sup_{z \in \partial\Omega} |\operatorname{Im} \psi_x(z)|$.

Remark 7.2.1. *Note that α_x increases as x approaches the boundary, so we can expect that the standard deviation of the error also increases as the point we are reconstructing approaches the boundary. In particular, in the current setting, we have $\|\psi_{(0,0)}\|_{L^\infty(\partial\Omega)} = 1/2$ and $\|\psi_x\|_{L^\infty(\partial\Omega)} = 2$ for $x \in \partial\Omega$.*

7.3 Approximate Bukhgeim solutions

In this chapter we make the first order approximation of the Bukhgeim solutions so that $u_{\lambda,x} = e^{i\lambda\psi_x}$ on the boundary. This scheme resembles the use of the Born approximation in the numerical reconstruction scheme given in [58]. Our goal is not to propose an alternative reconstruction scheme, but rather to study the possible numerical instabilities that could arise in Bukhgeim's reconstruction procedure as λ grows. Also we study how measurement error could propagate.

Approximating the standard Bukhgeim formula, we compute

$$\frac{\lambda}{\pi} \int_{\partial\Omega} (\Lambda_q - \Lambda_0) [e^{i\lambda\psi_x}] e^{i\lambda\bar{\psi}_x}$$

for different values of λ and potentials of conductivity type.

To compute the DtN map, we follow the strategy in [58] and consider radially symmetric conductivities. For radially symmetric conductivities $\gamma(x) = \gamma(|x|)$ and Ω the unit circle, the eigenfunctions of the DtN map are

$$\phi_n(\theta) = (2\pi)^{-1/2} e^{in\theta}.$$

For $\gamma = 1$ we have

$$\Lambda_1[\phi_n] = |n| \phi_n$$

and for other conductivities we have

$$\Lambda_\gamma[\phi_n] = \lambda_n \phi_n$$

with $\lambda_n \approx |n|$ for n large. From [58] we know that for two conductivities that satisfy $\gamma_L(x) \leq \gamma_U(x)$ the eigenvalues of their DtN maps satisfy $\lambda_n^L \leq \lambda_n^U$. Together with Lemma 4.1 in the same article (see below), we can compute the DtN map of a radial conductivity in the disk by approximating it from above and below. For conductivities equal to one close to the boundary we have $\Lambda_q = \Lambda_\gamma$.

Lemma 7.3.1. [58, Lemma 4.1] *Let $\Omega \in \mathbb{R}^2$ be the unit disc and $0 = r_0 < r_1 < \dots < r_{N-1} < r_N = 1$, where $N \geq 2$. Let $\gamma_j, j = 1, \dots, N$, be a collection of positive real numbers satisfying $\gamma_j \neq \gamma_{j+1}$ for $j = 1, \dots, N-1$, and assume $\gamma_N = 1$. Set $\gamma(r) = \gamma_j$ for $r_{j-1} \leq r < r_j, j = 1, \dots, N$. Then, the eigenvalues of Λ_γ are*

$$\lambda_n = |n| - 2|n|(1 + C_{n-1})^{-1}$$

where the values C_j are given recursively by $C_1 = \rho_1 r_1^{-2|n|}$ and

$$C_j = (\rho_j C_{j-1} + r_j^{-2|n|}) / (\rho_j + C_{j-1} r_j^{2|n|})$$

for $j = 2, \dots, N-1$, where $\rho_j = (\gamma_{j+1} + \gamma_j) / (\gamma_{j+1} - \gamma_j)$.

In Figure 7.1 and in Figure 7.2 we can see the approximate reconstruction formula for different frequencies in the case of two low contrast radially symmetric potentials. The figures show the true potential (blue) and the approximate formula (red) on the positive x-axis as a function of the radius. The approximate formula seems to come closer to the potential as the frequency increases but numerical instabilities show up for moderate frequency values ($\lambda = 25, 30$). As expected by Remark 7.2.1, numerical instabilities appear first close the boundary and then travel to the interior of the domain.

In Figure 7.3 we can see the approximate reconstruction formula for different frequencies for a potentials with higher contrast. The figures show

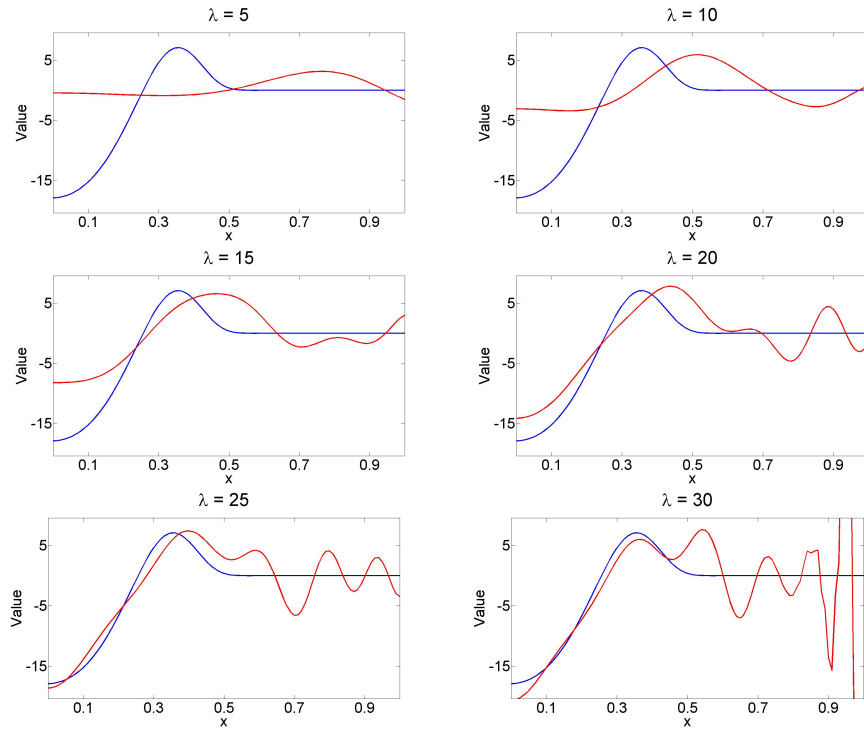


Figure 7.1: Approximation as the frequency increases.

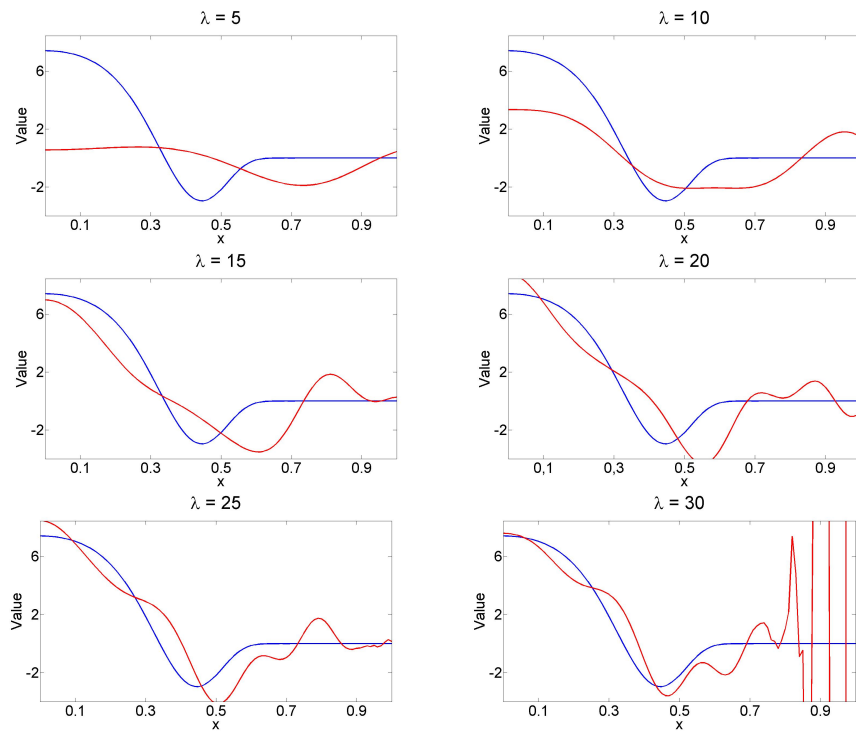


Figure 7.2: Approximation as the frequency increases.

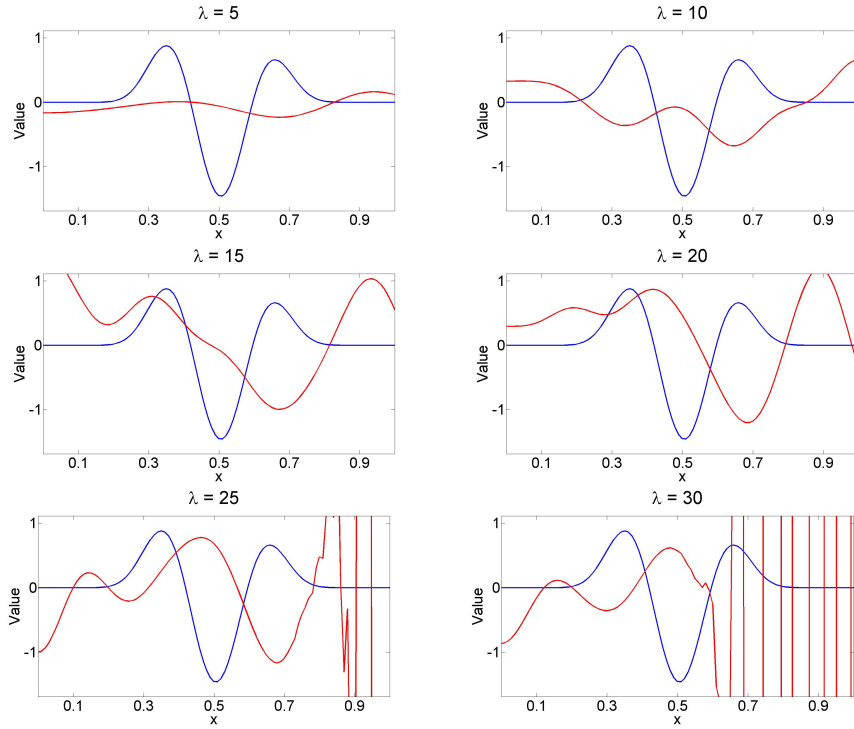


Figure 7.3: Approximation as the frequency increases.

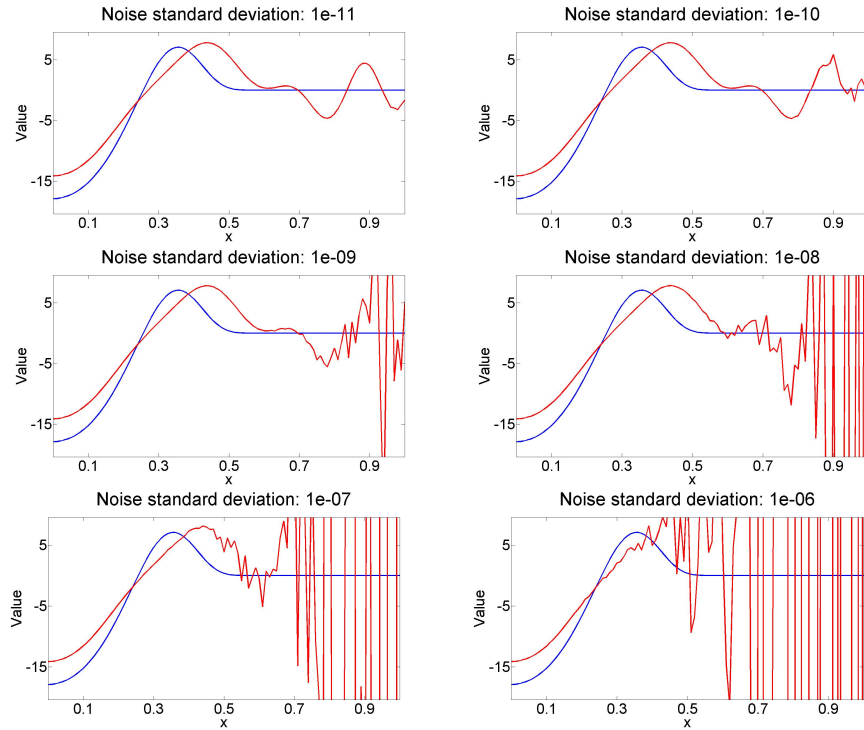


Figure 7.4: Approximation as the error in the measurements increases.

the true potential (blue) and the approximate formula (red) on the positive x-axis as a function of the radius. The approximate formula fails to converge to the potential in the frequencies considered and numerical instabilities are more significant. As expected by Remark 7.2.1, numerical instabilities appear first close the boundary and then travel to the interior of the domain.

Figure 7.4 shows the impact of measurement error in the approximation of a potential for $\lambda = 20$. The figure shows the true potential (blue) and the approximation (red) on the positive x-axis as a function of the radius. The noise incorporated in the measurements is a Gaussian white noise. We can see how small errors affect the approximation in a pronounced way.

7.4 Convergence of the main term

We compute the main term $T^\lambda[q]$ for different values of λ to study its convergence to the potential q . Given that $w_{\lambda,x}$ is considered a remainder, the study of the performance of this integral as an approximation to q could throw some light on the performance of the reconstruction q_λ .

The computation of these type of integrals is not trivial for large values of λ , as standard quadrature schemes are not viable for highly oscillating integrands. For example, the standard quadrature routine *quad2d* in the Matlab software package fails to converge for values of λ larger than 100 when q is taken to be the indicator of a square.

A standard approach to compute these type of integrals is the Filon method. This method relies on the fact that with P a polynomial and g an appropriate real-valued function, integrals of the form

$$\int_{\Omega} e^{i\lambda g} P$$

can be computed with closed form formulas. Any bounded function f is then approximated by a polynomial P so that

$$\left| \int_{\Omega} e^{i\lambda g} f - \int_{\Omega} e^{i\lambda g} P \right| \leq \|f - P\|_{L^1(\Omega)}.$$

See [55] for a detailed description of the method.

However, there are a number of parameters involved in the Filon method, such as the degree of the polynomials or the subdomains used to split the integration domain. To avoid the possibility of *contaminating* the results from a suboptimal choice of parameters, and given that there were no computational performance requirements, we decided to estimate the integrals by brute force, evaluating the integrand over a sufficiently dense mesh. We have used a uniform rectangular mesh, where the number of points varies between 640,000 and 64,000,000 depending on the value of λ . The main term has been computed on a coarser mesh, also regular, with 40,000 points. That is, the resolution of the images shown is of 40,000 pixels.

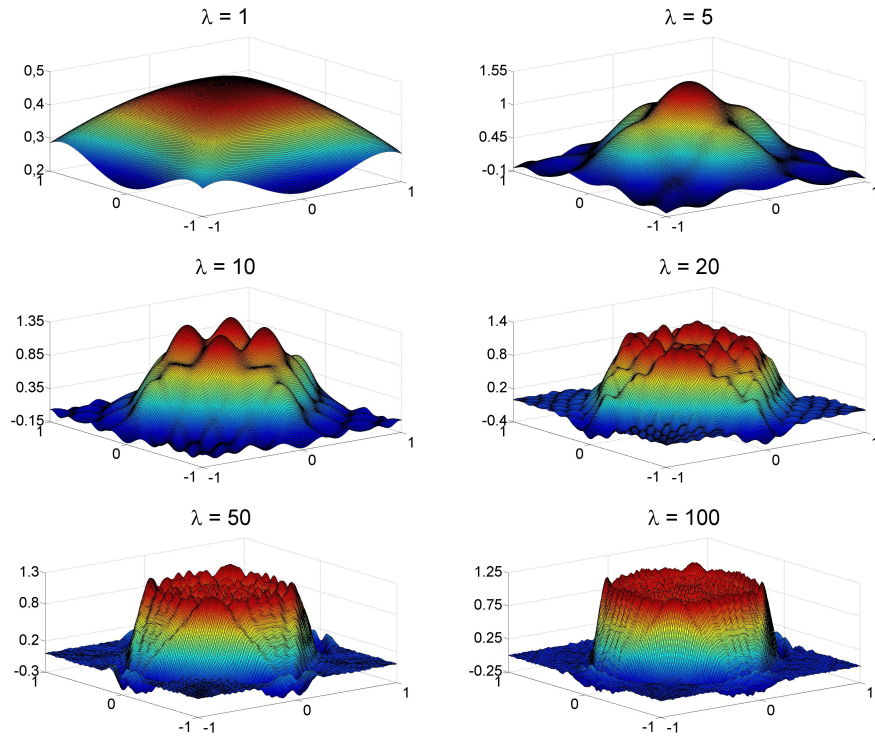


Figure 7.5: Disk indicator. Standard main term for different frequencies.

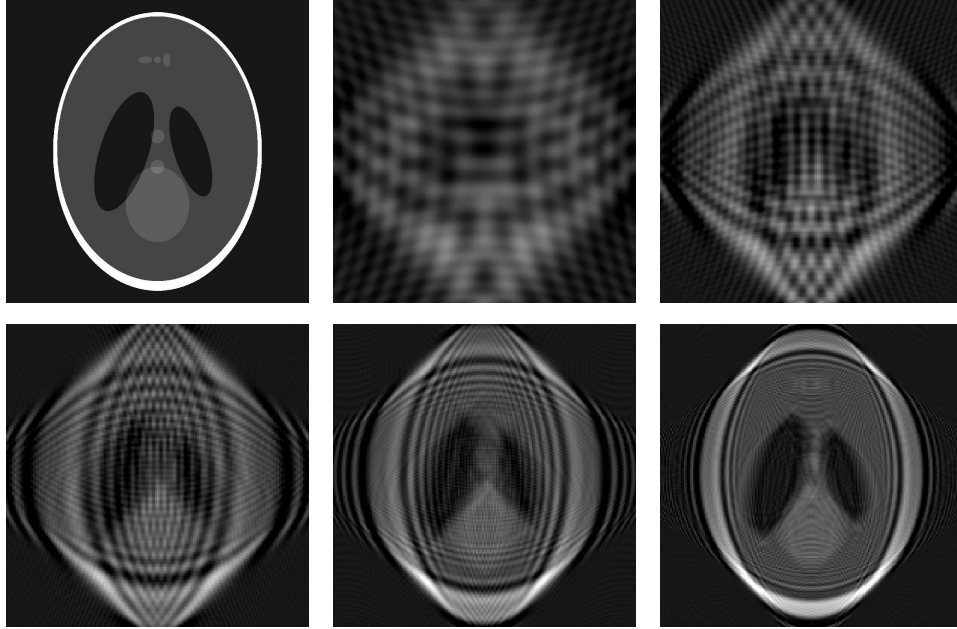


Figure 7.6: Shepp-Logan phantom. True potential and standard main term for $\lambda = 20, 50, 100, 200$ and 500 .

Figure 7.5 shows the main term for different values of λ when q is taken to be the indicator of the disk with radius $1/2$. We can see how the main term converges to the potential.

Figure 7.6 shows the main term for different values of λ when q is the Shepp-Logan phantom. We can see how the main term converges to the potential. Clearly the values of λ needed to have the main term close to the potential are higher than the ones required in Figure 7.5, where a simpler potential is considered.

7.5 Averaging

We study the effect of the averaging procedures introduced in Chapter 5 on the convergence of the main term to the potential. To compute the integrals we use the same approach and parameters as in Section 7.4.

7.5.1 Mollifier average

We consider the reconstruction formula introduced in Section 5.1

$$\lim_{\lambda \rightarrow \infty} \varphi_\sigma * BI_\lambda(x) = q(x)$$

where $\varphi_\sigma(x) := \sigma^{-2}\varphi(\sigma^{-1}x)$ with φ a positive mollifier supported in the unit ball, and

$$BI_\lambda(x) := \frac{\lambda}{\pi} \left\langle (\Lambda_q - \Lambda_0)[u_{\lambda,x}|_{\partial\Omega}], e^{i\lambda\bar{\psi}_x}|_{\partial\Omega} \right\rangle. \quad (7.1)$$

Recall that, using Alessandrini's identity (2.2), we have

$$\varphi_\sigma * BI_\lambda = \varphi_\sigma * T^\lambda[q] + \varphi_\sigma * T_w^\lambda[q]$$

where the first term in the right hand side is the main term which converges to the potential, and the second term is a remainder term which tends to zero. We study the convergence of the main term $\varphi_\sigma * T^\lambda[q]$, compared to the convergence of the main term in the standard Bukhgeim reconstruction formula $T^\lambda[q]$. We will use the mollifier given by

$$\varphi(z) = \begin{cases} e^{\frac{-1}{1-|z|^2}} & \text{for } |z| < 1 \\ 0 & \text{for } |z| \geq 1 \end{cases}$$

appropriately normalized so that its integral is equal to 1.

In the majority of the experiments considered the mollifier average has improved significantly the convergence of the main term to the potential for certain choices of the parameter σ . In Figure 7.16 we see the reduction of the error in the L^1 norm for different potentials and frequencies; the value

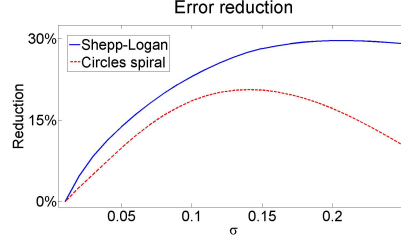


Figure 7.7: Error reduction in the mollifier average as a function of σ .

of σ used is different for each example. The most convenient value of σ in order to reduce the L^1 error depends on the potential and the frequency; we do not provide a criterion for its selection.

In Figure 7.7 we can see how the mollifier average reduces the error with respect to that of the main term. The reduction is shown as a function of the parameter σ . More precisely, in the figure we can see

$$f(\sigma) = 1 - \left\| q - \varphi_\sigma * T^\lambda[q] \right\|_{L^1} / \left\| q - T^\lambda[q] \right\|_{L^1}$$

where $\lambda = 50$ and $\sigma \in [0.01, 0.25]$. The domain where the *reconstruction* error is measured is the $[-1, 1] \times [-1, 1]$ square, where the support of the potentials is contained. The original potentials can be found in figures 7.13 and 7.15. We can see how the error reduction behaves differently for each potential; for the Shepp-Logan phantom we see a maximum reduction of 30% at $\sigma = 0.21$, while for the circles spirals the maximum reduction is 21% at $\sigma = 0.14$.

To use the mollifier average in practice we also have to take into account the effect of the boundary on the averaging. In the numerical experiments performed we have seen that the most convenient value of σ can lead to the support of the mollifier having diameter comparable to the size of the domain so that the support does not always fit inside the domain. In particular, for the example in Figure 7.7 with the Shepp-Logan phantom, 38% of the time. To overcome this difficulty we have simply computed the main term on the extended domain $[-1.25, 1.25] \times [-1.25, 1.25]$ and afterwards we compute the mollifier average in $[-1, 1] \times [-1, 1]$; the comparison between the methods is evaluated considering only the values inside $[-1, 1] \times [-1, 1]$.

In Figure 7.8 we can see the effect of the mollifier average on the Shepp-Logan phantom with $\lambda = 100$ for $\sigma = 0.07, 0.14$ (optimal) and 0.25; the error is reduced by 26%, 33% and 29% respectively. The effect of taking a suboptimal value for σ can be noticed in the images.

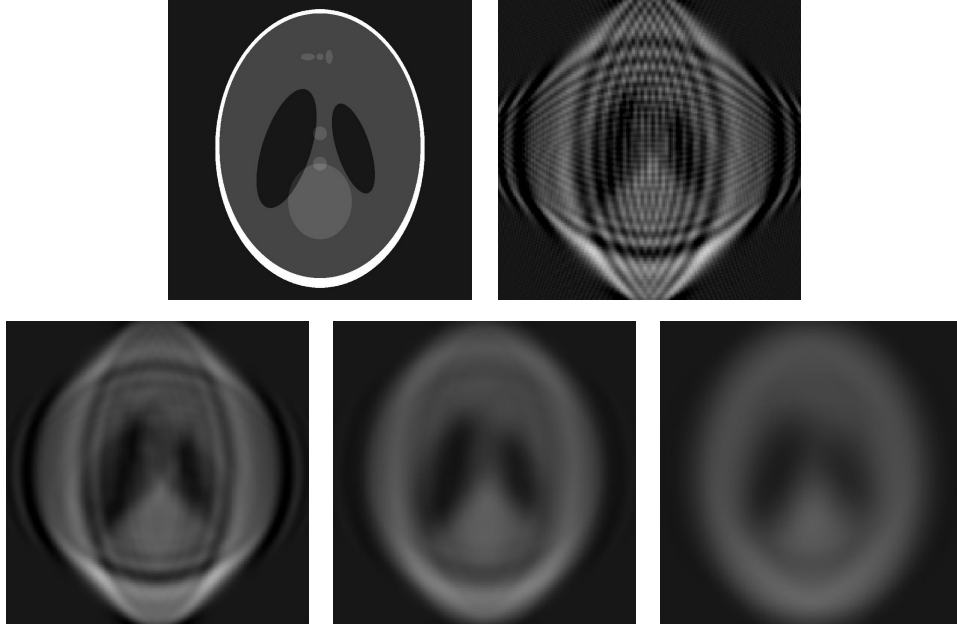


Figure 7.8: Shepp-Logan phantom $\lambda = 100$. Top: potential, standard main term. Bottom: mollifier average for $\sigma = 0.07, 0.14$ (optimal) and $\sigma = 0.25$.

7.5.2 Polar average

We recall briefly the reconstruction formula using the polar average introduced in Section 5.4. Letting

$$A_{\text{freq}}[F](\lambda) := \frac{1}{\lambda} \int_{\lambda}^{2\lambda} F(t) dt,$$

and $R_{x,\theta}(z)$ denote the rotation of z around x by an angle θ , the reconstruction formula with polar averaging reads

$$\lim_{\lambda \rightarrow \infty} A_{\text{freq}}^3[BI_x](\lambda) = q(x)$$

where

$$BI_x(\lambda) := \frac{\lambda}{2\pi^2} \int_0^{2\pi} \left\langle (\Lambda_q - \Lambda_0)[u_{\lambda,x,\theta}|_{\partial\Omega}], e^{i\lambda\bar{\psi}_x \circ R_{x,-\theta}}|_{\partial\Omega} \right\rangle d\theta \quad (7.2)$$

and the $u_{\lambda,x,\theta}$ are *rotated* Bukhgeim solutions for the Schrödinger equation.

Recall that, using Alessandrini's identity, we can decompose the reconstruction formula in terms of the angular averaging operator of Section 5.4

$$A_{\text{freq}}^3[BI_x](\lambda) = A_{\text{freq}}^3 \circ T^\lambda[A_{\text{ang}}[q]] + A_{\text{freq}}^3 \circ \frac{1}{2\pi} \int_0^{2\pi} T_{w_\lambda}^\lambda[q \circ R_\theta] d\theta. \quad (7.3)$$

where the first term is the main term which converges to the potential as λ tends to infinity, and the second term is a remainder which tends to zero as λ tends to infinity.

We study how the averaging operators A_{ang} and A_{freq} affect the convergence of the main term to the potential. Instead of studying the main term in (7.3), which involves the action of four operators, we consider three simpler cases:

- A frequency average $A_{\text{freq}} \circ T^\lambda[q]$.
- An angular average $T^\lambda[A_{\text{ang}}[q]]$.
- Both averages together $A_{\text{freq}} \circ T^\lambda[A_{\text{ang}}[q]]$.

Note that the angular average applied to the potential is a consequence of the average taken over rotated Bukhgeim solutions in the boundary information (7.2). Thus, the first of these cases, where only a frequency average is applied, corresponds to the use of a boundary information with *non-rotated* Bukhgeim solutions as in (7.1).

Experiments performed indicate that the angular average greatly improves the convergence of the main term to the potential. In other words, the angular average applied to the main term gives an approximation to the potential $T^\lambda \circ A_{\text{ang}}[q]$ which is significantly more accurate than the main term of the standard formula $T^\lambda[q]$ in all the experiments performed; see Figure 7.16. The performance of the frequency average is not so good, as the error reduction is not so significant and is not consistent across all the experiments performed.

Note that the recovery formulas (the standard formula and the ones involving averages) are asymptotic as λ tends to infinity. Thus, in practical terms, for the purpose of reconstruction we would always use the maximum possible value for λ . In this sense, introducing an average in frequency implies having to use suboptimal values of λ , as we need to use values smaller than the maximum possible. Following this reasoning, in the numerical experiments performed we compare frequency average in a band $[\lambda_{\min}, \lambda_{\max}]$ against the standard formula (or the angular average) with frequency λ_{\max} .

Another drawback in the use of the frequency average is that it is a parameter dependent method, as it depends on the size of the band for λ in which we perform the average. As numerical results obtained for this method have not been so exciting, we have not studied deeply the effects of taking bands with different sizes, or considered the use of weighted averages.

In Figure 7.9 we can see the effect of the polar average on the ovals for $\lambda = 20$, where the frequency average is performed for $\lambda \in [10, 20]$. In this example the frequency average reduces the L^1 error by a 1%, the angular average reduces the error by a 17% and the combination of both methods provides a reduction of a 11%. Thus, once applied the angular average, the frequency average increases the L^1 error. In the image we can see how the angular average has a significant positive effect.

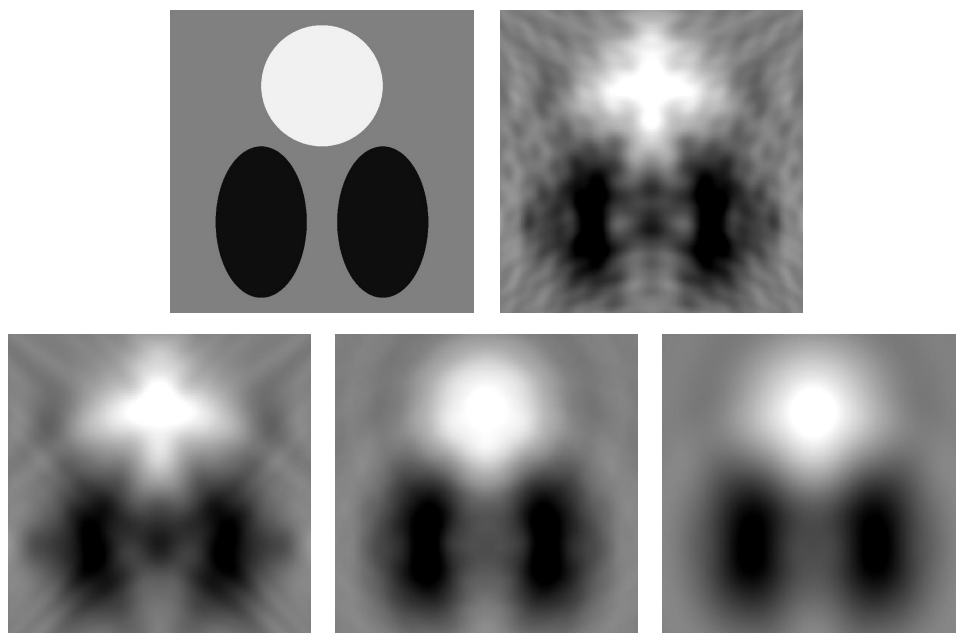


Figure 7.9: Ovals $\lambda = 20$. Top: potential, standard main term. Bottom: frequency average, angular average, angular and frequency average.

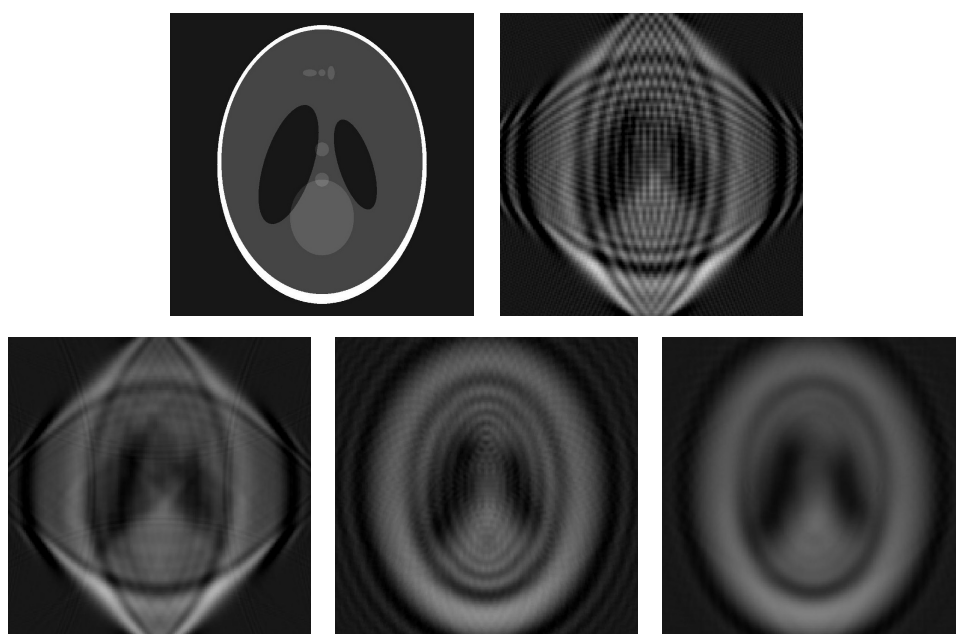


Figure 7.10: Shepp-Logan $\lambda = 100$. Top: potential, standard main term. Bottom: frequency average, angular average, angular and frequency average.

In Figure 7.10 we can see the effect of the polar average on the Shepp-Logan phantom for $\lambda = 100$, where the frequency average is performed for $\lambda \in [50, 100]$. In this setting the frequency average reduces the L^1 error by 18%, the angular average reduces the error by 20% and the combination of both methods provides a reduction of 24%. The angular average improves the image from standard formula significantly, and the image is further improved when we combine both averaging procedures. The improvement in the image from the use of the frequency averaging alone is not so evident.

7.5.3 Examples: Mollifier vs angular

We now compare the mollifier average with the angular average. To compute the mollifier average we use the value of σ which best reduces the L^1 norm error in each example. We see that both averaging procedures improve the L^1 error and the visual image obtained. The effect of both averaging procedures is significant.

We also consider a *combined averaging*, where we apply a mollifier average after having applied an angular average. Our experiments indicate that, for the best choice of σ , this combined method gives less error than the other two averaging procedures alone, but the extra improvement is not so pronounced. Note that for this combined averaging we also need to select a value for the parameter σ ; we follow the same strategy as before and we use the most convenient value for each example.

In Figures 7.11 to 7.15 we can see how the averaging procedures improve the convergence of the main term to the true potential. We have considered different geometries and frequencies.

In Figure 7.12 we see the effect of the mollifier and the angular average on the rectangles with frequency $\lambda = 10$. In the mollifier average we use $\sigma = 0.02$ while the combined method does not improve the error for any σ (compared to the angular average). The mollifier average reduces error less than 0.1% and there are no significant changes in the image (the value $\sigma = 0.02$ is small and the average hardly affects). The angular averaging reduces the error by 17% and the improvement can be noticed in the image.

In Figure 7.12 we see the effect of the mollifier and the angular average on the ovals with frequency $\lambda = 15$. In the mollifier average we use $\sigma = 0.24$ and in the combined average we use $\sigma = 0.20$. The error reduction is 9% for the mollifier, 13% for the angular and 15% for the combined.

In Figure 7.13 we see the effect of the mollifier and the angular average on the circles spiral with frequency $\lambda = 30$. In the mollifier average we use $\sigma = 0.22$ and in the combined average we use $\sigma = 0.13$. The error reduction is 20% for the mollifier, 23% for the angular and 24% for the combined.

In Figure 7.14 we see the effect of the mollifier and the angular average with frequency $\lambda = 50$. In the mollifier average we use $\sigma = 0.15$ and in the combined average we use $\sigma = 0.13$. The error reduction is 18% for the

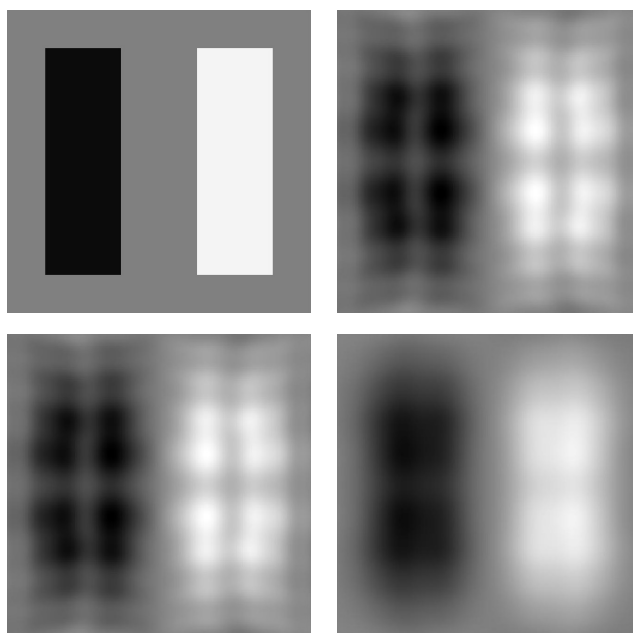


Figure 7.11: Rectangles $\lambda = 10$. Top: potential, standard main term.
Bottom: mollifier average, angular average.

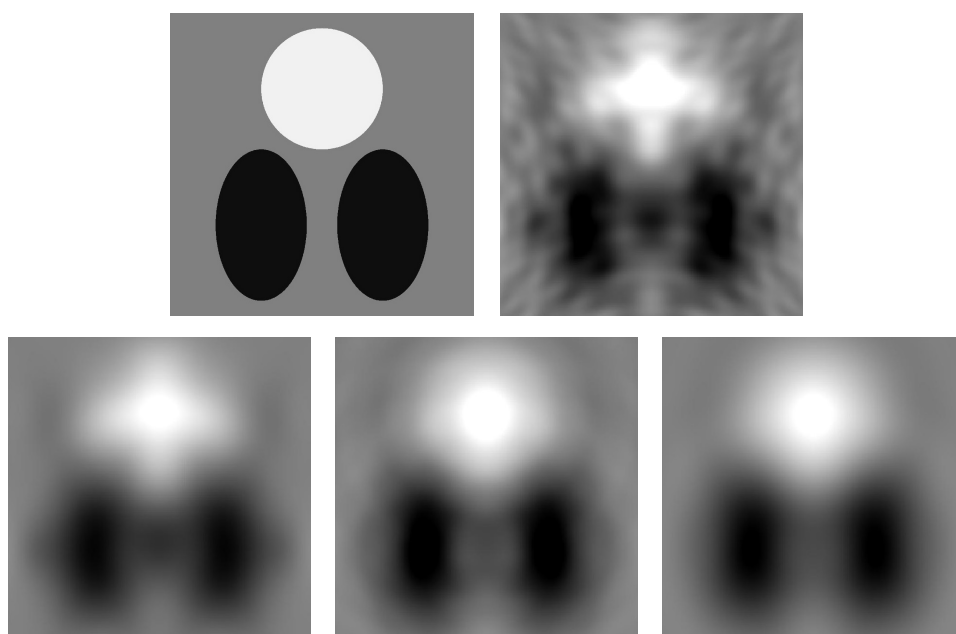


Figure 7.12: Ovals $\lambda = 15$. Top: potential, standard main term.
Bottom: mollifier average, angular average, angular and mollifier average.

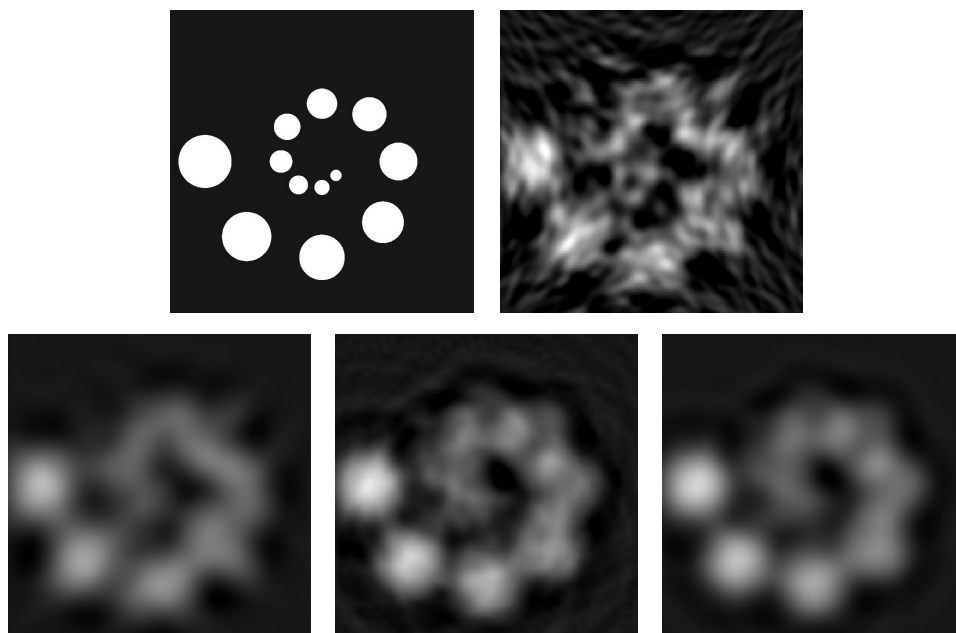


Figure 7.13: Circles spiral $\lambda = 30$. Top: potential, standard main term. Bottom: mollifier average, angular average, angular and mollifier average.

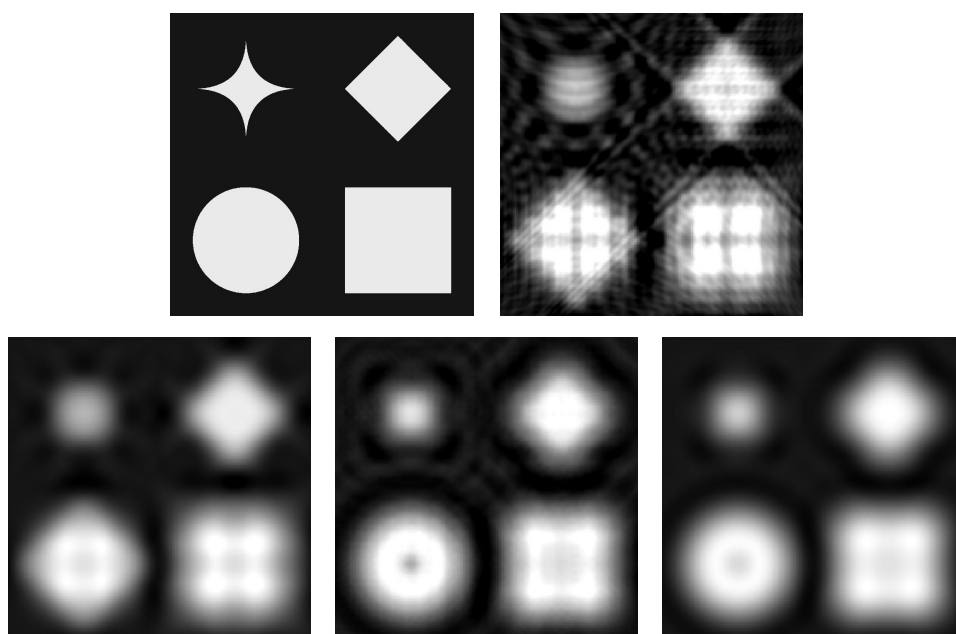


Figure 7.14: Geometric figures $\lambda = 50$. Top: potential, standard main term. Bottom: mollifier average, angular average, angular and mollifier average.

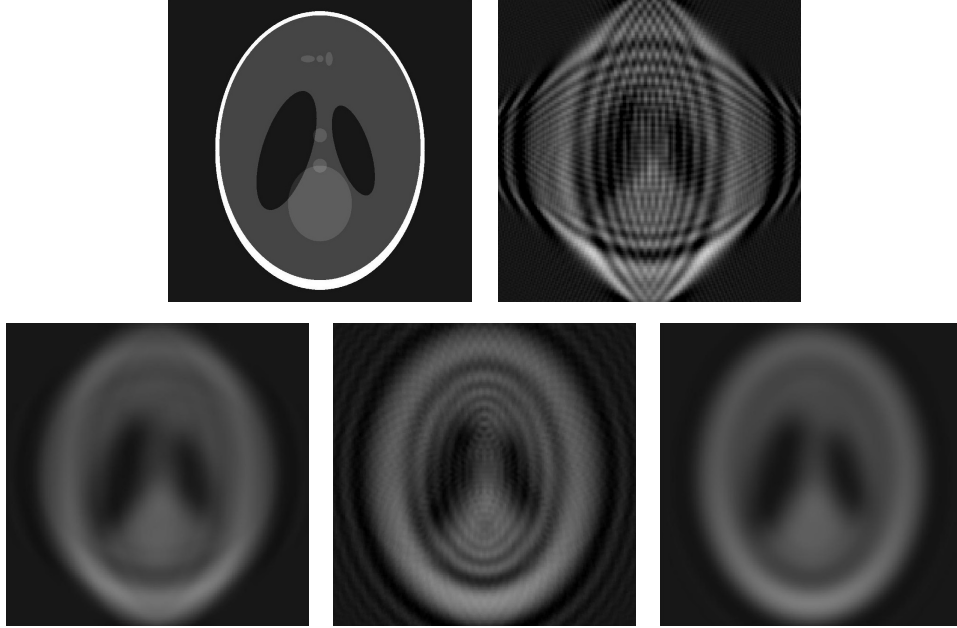


Figure 7.15: Shepp-Logan phantom $\lambda = 100$. Top: potential, standard main term. Bottom: mollifier average, angular average, angular and mollifier average.

mollifier and for the angular and 21% for the combined.

In Figure 7.15 we see the effect of the mollifier and the angular average on the Shepp-Logan phantom with frequency $\lambda = 100$. In the mollifier average and in the combined average we use $\sigma = 0.14$. The error reduction is 33% for the mollifier, 20% for the angular and 34% for the combined.

In Figure 7.16 we can compare the error reductions in the L^1 norm for the mollifier average, the angular average and the combination of the angular and the mollifier average. The table contains the reduction for the examples previously described and for the same potentials with other choices of the frequency parameter λ ; we see that results are stable in this sense. It is interesting that the effect of the averaging procedures seems to be more pronounced as frequency increases.

7.6 Conclusions

Numerical experiments performed suggest that the convergence rate of the standard formula for the Bukhgeim method could be too slow for some practical applications, but the averaging procedures discussed could make the difference.

In order to obtain reasonably accurate reconstructions, the standard Bukhgeim method seems to require values λ which could be too large to avoid numerical instabilities. Note that the parameter λ controls both the

	λ	Mollifier	Angular	Combined
Rectangles	10	0%	17%	17%
Rectangles	15	1%	16%	16%
Rectangles	20	12%	22%	24%
Rectangles	30	18%	27%	28%
Ovals	15	9%	13%	15%
Ovals	20	13%	17%	18%
Ovals	30	12%	17%	19%
Circles spiral	20	23%	20%	26%
Circles spiral	30	20%	23%	24%
Circles spiral	50	21%	24%	26%
Geometric figures	20	7%	14%	14%
Geometric figures	30	11%	13%	15%
Geometric figures	50	18%	18%	21%
Shepp-Logan phantom	50	30%	19%	30%
Shepp-Logan phantom	75	31%	18%	32%
Shepp-Logan phantom	100	33%	20%	34%

Figure 7.16: Error reduction in the L^1 norm.

amplitude and the frequency of the solutions, and when these are too large, the error of the numerical integration at the boundary becomes large as well.

Numerical difficulties as the frequency of the solutions increases have also been observed in other reconstruction procedures. For example in [42], where Knudsen develops a numerical algorithm for reconstructing a conductivity in the plane, it is pointed out that the scattering transform becomes highly inaccurate for $|k| > 25$. Similar inaccuracies are also present in other numerical methods relying on CGO¹ (e.g. [10, 49, 58]), where $|k|$ is never taken larger than 35. Values for the frequency parameter k used in real-life measurement studies are considerably lower (e.g. [37, 38]) for similar reasons.

However, the averaging procedures introduced have proven to be useful for improving the convergence of the main term in the Bukhgeim approach and the images of the potentials obtained are clearly enhanced. In our examples, the angular averages outperformed the mollifier averaging procedure at lower frequencies. On the other hand, the artefacts introduced by the standard Bukhgeim formula are removed more effectively by the mollifier procedure rather than the angular. Both procedures improved the contrast

¹Note that within the unit ball, the maximum amplitude of Bukhgeim's solutions for $\lambda = 15$ is the same as the amplitude of CGO solutions with $k = 30$.

of the images and reduced the L^1 significantly, and these improvements are more pronounced when both procedures are combined.

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